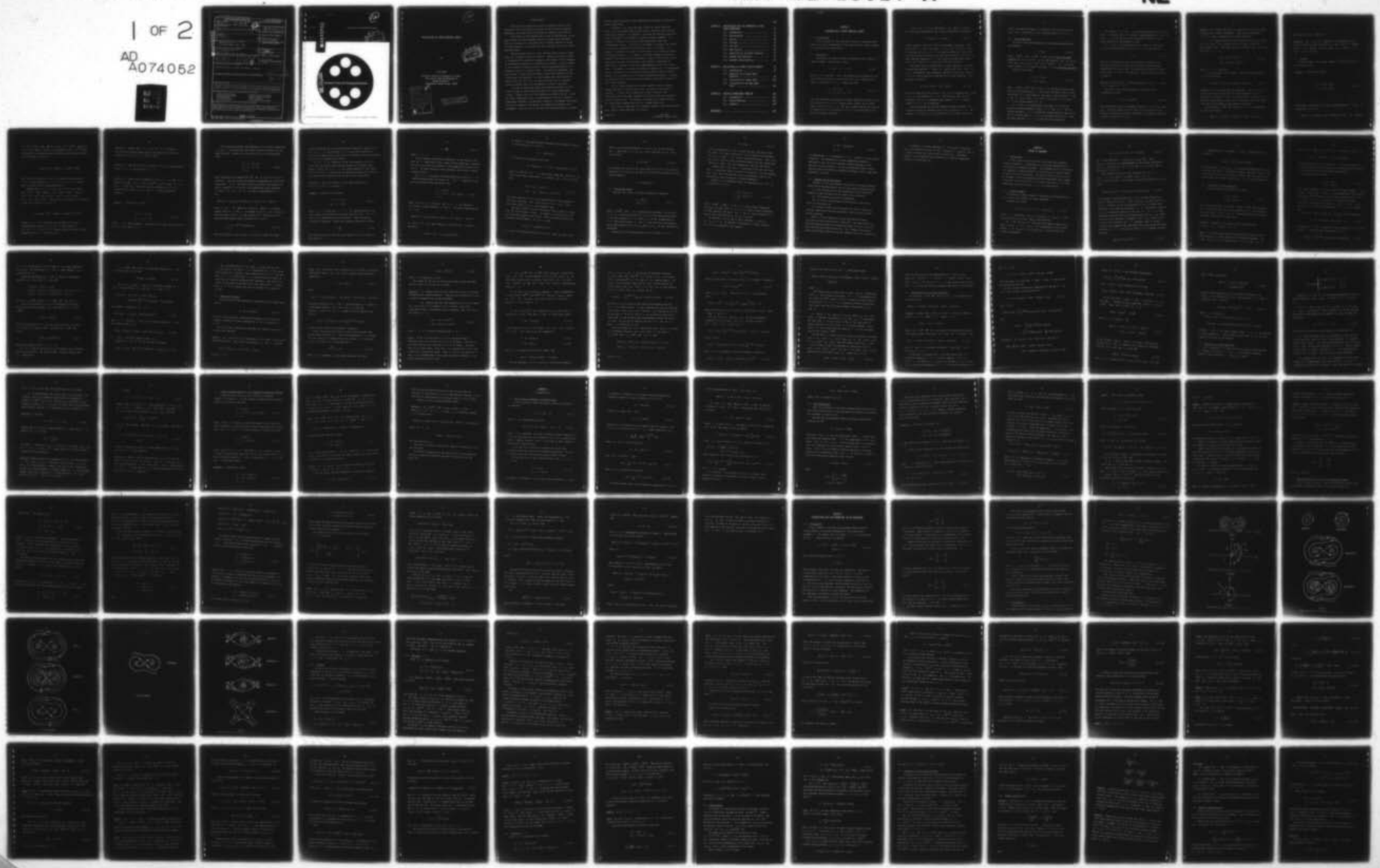
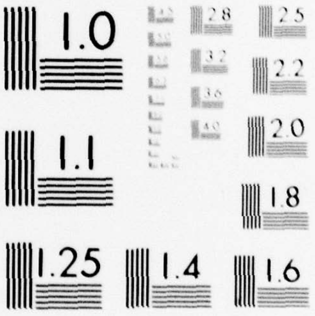
AD-A074 052   BROWN UNIV PROVIDENCE R I LEFSCHETZ CENTER FOR DYNAM--ETC   F/G 12/1
APPLICATIONS OF CENTRE MANIFOLD THEORY.(U)
JUN 79   J CARR   DAAG29-76-G-0294

UNCLASSIFIED   ARO-13915.17-M   NL

1 OF 2
AD
A074052




1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A074052

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
(11) 13915.17-M (18) ARO

2. GOVT ACCESSION NO.

(12)

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
(6) Applications of Centre Manifold Theory.

5. TYPE OF REPORT & PERIOD COVERED
(9) Technical rept.,

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
(10) Jack/Carr

8. CONTRACT OR GRANT NUMBER(s)
(15) DAAG29-76-G-0294

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Brown University
Providence, Rhode Island 02912

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709

12. REPORT DATE
(11) June 1979

13. NUMBER OF PAGES
131

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
(12) 137 p.

LEVEL II

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DDC
RECEIVED
SEP 19 1979
C

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

centre manifold theory
differential equations
manifold theory
bifurcation theory
dynamic bifurcation theory
Liapunov-Schmitt procedure
static solutions
finite dimensional systems

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The purpose of the lectures was to give an introduction to the applications of centre manifold theory to differential equations. Most of the material is presented in an informal fashion, by means of worked examples in the hope that this clarifies the use of centre manifold theory.

401 834

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DDC FILE COPY

Lefschetz Center for Dynamical Systems

⑫

APPLICATIONS OF CENTRE MANIFOLD THEORY

DDC
RECEIVED
SEP 19 1979
C

BY

JACK CARR

LEFSCHETZ CENTER FOR DYNAMICAL SYSTEMS
DIVISION OF APPLIED MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RHODE ISLAND 02912

Accession For
NTIS GRA&I ☑
DDC TAB ☐
Unannounced ☐
Justification
By
Distribution/
Availability Codes
Dist A
Avail and/or special

This document has been approved for public release and sale; its distribution is unlimited.

PREFACE

These notes are based on a series of lectures given in the Lefschetz Center for Dynamical Systems in the Division of Applied Mathematics at Brown University during the academic year 1978-79.

The purpose of the lectures was to give an introduction to the applications of centre manifold theory to differential equations. Most of the material is presented in an informal fashion, by means of worked examples in the hope that this clarifies the use of centre manifold theory.

The main application of centre manifold theory given in these notes is to dynamic bifurcation theory. Dynamic bifurcation theory is concerned with topological changes in the nature of the solutions of differential equations as parameters are varied. Such an example is the creation of periodic orbits from an equilibrium point as a parameter crosses a critical value. In certain circumstances, the application of centre manifold theory reduces the dimension of the system under investigation. In this respect the centre manifold theory plays the same role for dynamic problems as the Liapunov-Schmitt procedure plays for the analysis of static solutions.

In order to make these notes more widely accessible, we give a full account of centre manifold theory for finite dimensional systems. Indeed, the first five chapters are devoted to this. Once the finite dimensional case is understood, the step up to infinite dimensional problems is essentially technical. Throughout these notes we give the simplest such theory, for example our equations are autonomous. Once the core of an idea has been understood in a simple setting, generalizations to more complicated situations are much more readily understood.

In Chapter 1, we state the main results of centre manifold theory for finite dimensional systems and we illustrate their use by a few simple examples. In Chapter 2, we prove the theorems which were stated in Chapter 1, and Chapter 3 contains further examples. In Section 2 of Chapter 3 we outline Hopf bifurcation theory for 2-dimensional systems. In Section 3 of Chapter 3 we apply this theory to a singular perturbation problem which arises in biology. In Example 3 of Chapter 6 we apply the same theory to a system of partial differential equations. In Chapter 4 we study a dynamic bifurcation problem in the plane with two parameters. Some of the results in this Chapter are new and, in particular, they confirm a conjecture of Takens [49]. *Chapter 4 can be read independently of the rest of the notes.* In Chapter 5, we apply the theory of Chapter 4 to a 4-dimensional system. In Chapter 6, we extend the centre manifold theory given in Chapter 2 to a simple class of infinite dimensional problems. Finally, we illustrate their use in partial differential equations by means of some simple examples.

I first became interested in centre manifold theory through reading Dan Henry's Lecture Notes [30]. My debt to these notes is enormous. I would like to thank Jack K. Hale, Dan Henry and John Mallet-Paret for many valuable discussions during the gestation period of these notes. In addition, my thanks go to Sandra Spinacci for converting my manuscript into this elegant typescript.

This work was done with the financial support of the United States Army, Durham, under AROD DAAG 29-76-G0294.

June 1979

Jack Carr
Providence, Rhode Island

PAGE

CHAPTER 4. BIFURCATIONS WITH TWO PARAMETERS IN TWO SPACE DIMENSIONS 57

4.1. INTRODUCTION 57
4.2. PRELIMINARIES 59
4.3. SCALING 66
4.4. THE CASE $\varepsilon_1 > 0$ 67
4.5. THE CASE $\varepsilon_1 < 0$ 82
4.6. MORE SCALING 84
4.7. COMPLETION OF THE PHASE PORTRAITS ... 87
4.8. REMARKS AND EXERCISES 87
4.9. QUADRATIC NONLINEARITIES 89

CHAPTER 5. APPLICATION TO A PANEL FLUTTER PROBLEM 94
5.1. INTRODUCTION 94
5.2. REDUCTION TO A SECOND ORDER EQUATION 95
5.3. CALCULATION OF LINEAR TERMS 100
5.4. CALCULATION OF THE NONLINEAR TERMS 101

CHAPTER 6. INFINITE DIMENSIONAL PROBLEMS 104
6.1. INTRODUCTION 104
6.2. CENTRE MANIFOLDS 104
6.3. EXAMPLES 108

REFERENCES 128

CHAPTER 1

INTRODUCTION TO CENTRE MANIFOLD THEORY

1.1. Introduction

In this chapter we state the main results of centre manifold theory for finite dimensional systems and give some simple examples to illustrate their application.

1.2. Motivation

To motivate the study of centre manifolds we first look at a simple example. Consider the system

$$\dot{x} = ax^3, \quad \dot{y} = -y + y^2, \tag{1.2.1}$$

where a is a constant. Since the equations are uncoupled we can easily show that the zero solution of (1.2.1) is asymptotically stable if and only if $a < 0$. Suppose now that

$$\begin{aligned} \dot{x} &= ax^3 + x^2y \\ \dot{y} &= -y + y^2 + xy - x^3. \end{aligned} \tag{1.2.2}$$

Since the equations are coupled we cannot immediately decide if the zero solution of (1.2.2) is asymptotically stable, but we might suspect that it is if $a < 0$. The key to understanding the relation of equation (1.2.2) to equation (1.2.1) is an abstraction of the idea of uncoupled equations.

A curve, say $y = h(x)$, defined for $|x|$ small, is said to be an invariant manifold for the system of differential equations

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y), \tag{1.2.3}$$

if the solution $x(t),y(t)$ of (1.2.3) through $(x_0,h(x_0))$ lies on the curve $y = h(x)$ for small t, i.e., $y(t) = h(x(t))$. Thus, for equation (1.2.1), $y = 0$ is an invariant manifold. Note that in deciding upon the stability of the zero solution of (1.2.1), the only important equation is $\dot{x} = ax^3$, that is we only need study a first order equation on a particular invariant manifold.

The theory that we develop tells us that equation (1.2.3) has an invariant manifold $y = h(x)$, $|x|$ small, with $h(x) = O(x^2)$ as $x \to 0$. Furthermore, the asymptotic stability of the zero solution of (1.2.2) can be proved by studying a first order equation. This equation is given by

$$\dot{u} = au^3 + u^2h(u) = au^3 + O(u^4), \tag{1.2.4}$$

and we see that the zero solution of (1.2.4) is asymptotically stable if $a < 0$ and unstable if $a > 0$. This tells us that the zero solution of (1.2.2) is asymptotically stable if $a < 0$ and unstable if $a > 0$ as we expected.

We are also able to use this method to obtain estimates for the rate of decay of solutions of (1.2.2) in the case $a < 0$. For example, if $x(t),y(t)$ is a solution of (1.2.2) with $x(0),y(0)$

small, we prove that there is a solution $u(t)$ of (1.2.4) such that $x(t) = u(t)(1+o(1))$, $y(t) = h(u(t))(1+o(1))$ as $t \to \infty$.

1.3. Centre Manifolds

We first recall the definition of an invariant manifold for the equation

$$\dot{x} = N(x) \tag{1.3.1}$$

where $x \in \mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is said to be a local invariant manifold for (1.3.1) if for $x_0 \in S$, the solution $x(t)$ of (1.3.1) with $x(0) = x_0$ is in S for $|t| < T$ where $T > 0$. If we can always choose $T = \infty$, then we say that S is an invariant manifold.

Consider the system

$$\begin{aligned} \dot{x} &= Ax + f(x,y) \\ \dot{y} &= By + g(x,y) \end{aligned} \tag{1.3.2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and A and B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts. The functions f and g are C^2 with $f(0,0) = 0$, $f'(0,0) = 0$, $g(0,0) = 0$, $g'(0,0) = 0$ (f' is the Jacobian matrix of f).

If f and g are identically zero then (1.3.2) has two obvious invariant manifolds, namely $x = 0$ and $y = 0$. The invariant manifold $x = 0$ is called the stable manifold, since if we restrict initial data to $x = 0$, all solutions tend to zero. The invariant manifold $y = 0$ is called the centre manifold.

In general, if $y = h(x)$ is an invariant manifold for (1.3.2) and h is smooth, then it is called a centre manifold if $h(0) = 0$, $h'(0) = 0$. We use the term centre manifold in place of local centre manifold if the meaning is clear.

If f and g are identically zero, then all solutions of (1.3.2) tend exponentially fast, as $t \to \infty$, to solutions of

$$\dot{x} = Ax, \tag{1.3.3}$$

that is, the equation on the centre manifold determines the asymptotic behavior of solutions of the full equation modulo exponentially decaying terms. We now give the analogue of these results when f and g are non-zero. These results are proved in Chapter 2.

Theorem 1. There exists a centre manifold for (1.3.2), $y = h(x)$, $|x| < \delta$, where h is C^2.

The flow on the centre manifold is governed by the n-dimensional system

$$\dot{u} = Au + f(u,h(u)) \tag{1.3.4}$$

which generalizes the corresponding problem (1.3.3) for the linear case. The next theorem tells us that (1.3.4) contains all the necessary information needed to determine the asymptotic behavior of small solutions of (1.3.2).

Theorem 2 (a) Suppose that the zero solution of (1.3.4) is stable (asymptotically stable) (unstable). Then the zero solution of (1.3.2) is stable (asymptotically stable)(unstable).

(b) Suppose that the zero solution of (1.3.4) is stable. Let $(x(t),y(t))$ be a solution of (1.3.2) with $(x(0),y(0))$ sufficiently small. Then there exists a solution $u(t)$ of (1.3.4) such that as $t \to \infty$,

$$\begin{aligned} x(t) &= u(t) + O(e^{-\gamma t}) \\ y(t) &= h(u(t)) + O(e^{-\gamma t}) \end{aligned} \qquad (1.3.5)$$

where $\gamma > 0$ is a constant.

If we substitute $y(t) = h(x(t))$ into the second equation in (1.3.2) we obtain

$$h'(x)[Ax + f(x,h(x))] = Bh(x) + g(x,h(x)). \qquad (1.3.6)$$

Equation (1.3.6) together with the conditions $h(0) = 0$, $h'(0) = 0$ is the system to be solved for the centre manifold. This is impossible, in general, since it is equivalent to solving (1.3.2). The next result however, shows that in principle, the centre manifold can be approximated to any degree of accuracy.

For functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$ which are C^1 in a neighborhood of the origin define

$$(M\phi)(x) = \phi'(x)[Ax + f(x,\phi(x))] - B\phi(x) - g(x,\phi(x)).$$

Note that by (1.3.6), $(Mh)(x) = 0$.

Theorem 3. Let ϕ be a C^1 mapping of a neighborhood of the origin in $\mathbb{R}^n$ into $\mathbb{R}^m$ with $\phi(0) = 0$ and $\phi'(0) = 0$. Suppose that as $x \to 0$, $(M\phi)(x) = O(|x|^q)$ where $q > 1$. Then as $x \to 0$, $|h(x) - \phi(x)| = O(|x|^q)$.

1.4. Examples

We now consider a few simple examples to illustrate the use of the above results.

Example 1. Consider the system

$$\begin{aligned} \dot{x} &= xy + ax^3 + by^2x \\ \dot{y} &= -y + cx^2 + dx^2y. \end{aligned} \tag{1.4.1}$$

By Theorem 1, equation (1.4.1) has a centre manifold $y = h(x)$. To approximate h we set

$$(M\phi)(x) = \phi'(x)[x\phi(x) + ax^3 + bx\phi^2(x)] + \phi(x) - cx^2 - dx^2\phi(x).$$

If $\phi(x) = O(x^2)$ then $(M\phi)(x) = \phi(x) - cx^2 + O(x^4)$. Hence, if $\phi(x) = cx^2$, $(M\phi)(x) = O(x^4)$, so by Theorem 3, $h(x) = cx^2 + O(x^4)$. By Theorem 2, the equation which determines the stability of the zero solution of (1.4.1) is

$$\dot{u} = uh(u) + au^3 + buh^2(u) = (a+c)u^3 + O(u^5).$$

Thus the zero solution of (1.4.1) is asymptotically stable if $a + c < 0$ and unstable if $a + c > 0$. If $a + c = 0$ then we have to obtain a better approximation to h.

Suppose that $a + c = 0$. Let $\phi(x) = cx^2 + \psi(x)$ where $\psi(x) = O(x^4)$. Then $(M\phi)(x) = \psi(x) - cdx^4 + O(x^6)$. Thus, if $\phi(x) = cx^2 + cdx^4$ then $(M\phi)(x) = O(x^6)$ so by Theorem 3, $h(x) = cx^2 + cdx^4 + O(x^6)$. The equation that governs the stability of the zero solution of (1.4.1) is

$$\dot{u} = uh(u) + au^3 + buh^2(u) = (cd+bc^2)u^5 + O(u^7).$$

Hence, if $a + c = 0$, then the zero solution of (1.4.1) is asymptotically stable if $cd + bc^2 < 0$ and unstable if $cd + bc^2 > 0$. If $cd + bc^2 = 0$ then we have to obtain a better approximation to h (see Exercise 1).

Exercise 1. Suppose that $a + c = cd + bc^2 = 0$ in Example 1. Show that the equation which governs the stability of the zero solution of (1.4.1) is $\dot{u} = -cd^2u^7 + O(u^9)$.

Exercise 2. Show that the zero solution of (1.2.2) is asymptotically stable if $a \leq 0$ and unstable if $a > 0$.

Exercise 3. Suppose that in equation (1.3.2), $n = 1$ so that $A = 0$. Suppose also that $f(x,y) = ax^p + O(|x|^{p+1} + |y|^q)$ where $2q \geq p + 1$ and a is non-zero. Show that the zero solution of (1.3.2) is asymptotically stable if $a < 0$ and p is odd, and unstable otherwise.

Example 2. Consider the system

$$\begin{aligned} \dot{x} &= \varepsilon x - x^3 + xy \\ \dot{y} &= -y + y^2 - x^2 \end{aligned} \qquad (1.4.2)$$

where ε is a real parameter. The object is to study small solutions of (1.4.2) for small $|\varepsilon|$.

The linearized problem corresponding to (1.4.2) has eigenvalues -1 and ε. This means that the results given in Section 3 do not apply directly. However, we can write (1.4.2) in the equivalent form

$$\begin{aligned} \dot{x} &= \varepsilon x - x^3 + xy \\ \dot{y} &= -y + y^2 - x^2 \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{1.4.3}$$

When considered as an equation on $\mathbb{R}^3$ the εx term in (1.4.3) is nonlinear. Thus the linearized problem corresponding to (1.4.3) has eigenvalues $-1,0,0$. The theory given in Section 3 now applies so that by Theorem 1, (1.4.3) has a two dimensional centre manifold $y = h(x,\varepsilon)$, $|x| < \delta_1$, $|\varepsilon| < \delta_2$. To find an approximation to h set

$$(M\phi)(x,\varepsilon) = \phi_x(x,\varepsilon)[\varepsilon x - x^3 + x\phi(x,\varepsilon)] + \phi(x,\varepsilon) + x^2 - \phi^2(x,\varepsilon).$$

Then, if $\phi(x) = -x^2$, $(M\phi)(x,\varepsilon) = O(C(x,\varepsilon))$ where C is a homogeneous cubic in x and ε. By Theorem 3, $h(x,\varepsilon) = -x^2 + O(C(x,\varepsilon))$. Note also that $h(0,\varepsilon) = 0$ (see Section 2.6). By Theorem 2 the equation which governs small solutions of (1.4.3) is

$$\begin{aligned} \dot{u} &= \varepsilon u - 2u^3 + O(|u|C(u,\varepsilon)) \\ \dot{\varepsilon} &= 0 . \end{aligned} \tag{1.4.4}$$

The zero solution $(u,\varepsilon) = (0,0)$ of (1.4.4) is stable for small ε

so the representation of solutions given by Theorem 2 applies here. For $-\delta_2 < \varepsilon < 0$ the solution $u = 0$ of the first equation in (1.4.4) is asymptotically stable and so by Theorem 2 the zero solution of (1.4.2) is asymptotically stable.

For $0 < \varepsilon < \delta_2$, solutions of the first equation in (1.4.4) consist of two orbits connecting the origin to two small fixed points. Hence, for $0 < \varepsilon < \delta_2$ the stable manifold of the origin for (1.4.2) forms a separatrix, the unstable manifold consisting of two stable orbits connecting the origin to the fixed points.

Exercise 4. Study the behavior of all small solutions of $\ddot{w} + \dot{w} + \varepsilon w + w^3 = 0$ for small ε.

Example 3. Consider the equations

$$\begin{aligned} \dot{y} &= -y + (y+c)z \\ \varepsilon\dot{z} &= y - (y+1)z \end{aligned} \tag{1.4.5}$$

where $\varepsilon > 0$ is small and $0 < c < 1$. The above equations arise from a model of the kinetics of enzyme reactions [29]. If $\varepsilon = 0$, then (1.4.5) degenerates into one algebraic equation and one differential equation. Solving the algebraic equation we obtain

$$z = \frac{y}{y+1} \tag{1.4.6}$$

and substituting this into the first equation in (1.4.5) leads to the equation

$$\dot{y} = \frac{-\lambda y}{1+y} \qquad (1.4.7)$$

where $\lambda = 1 - c$.

Using singular perturbation techniques, it was shown in [29] that for ε sufficiently small, under certain conditions, solutions of (1.4.5) are close to solutions of the degenerate system (1.4.6), (1.4.7). We shall show how centre manifolds can be used to obtain a similar result.

Let $t = \varepsilon\tau$. We denote differentiation with respect to t by $\cdot$ and differentiation with respect to τ by $'$. Equation (1.4.5) can be rewritten in the equivalent form

$$\begin{aligned} y' &= \varepsilon f(y,w) \\ w' &= -w + y^2 - yw + \varepsilon f(y,w) \qquad (1.4.8) \\ \varepsilon' &= 0 \end{aligned}$$

where $f(y,w) = -y + (y+c)(y-w)$ and $w = y - z$. By Theorem 1, (1.4.8) has a centre manifold $w = h(y,\varepsilon)$. To find an approximation to h set

$$(M\phi)(y,\varepsilon) = \varepsilon\phi_y(y,\varepsilon)f(y,w) + h(y,\varepsilon) - y^2 + yh(y,\varepsilon) - \varepsilon f(y,w).$$

If $\phi(y,\varepsilon) = y^2 - \lambda\varepsilon y$ then $(M\phi)(y,\varepsilon) = O(|y|^3+|\varepsilon|^3)$ so that by Theorem 3,

$$h(y,\varepsilon) = y^2 - \lambda\varepsilon y + O(|y|^3+|\varepsilon|^3).$$

By Theorem 2, the equation which determines the asymptotic behavior of small solutions of (1.4.8) is

$$u' = \varepsilon f(u,h(u,\varepsilon))$$

or in terms of the original time scale

$$\dot{u} = f(u,h(u,\varepsilon)) = -\lambda(u-u^2) + O(|\varepsilon u|+|u|^3). \quad (1.4.9)$$

Again, by Theorem 2, if ε is sufficiently small and $y(0),z(0)$ are sufficiently small, then there is a solution $u(t)$ of (1.4.9) such that

$$\begin{aligned} y(t) &= u(t) + O(e^{-t/\varepsilon}) \\ z(t) &= y(t) - h(y(t),\varepsilon) + O(e^{-t/\varepsilon}). \end{aligned} \quad (1.4.10)$$

Note that equation (1.4.7) is an approximation to the equation on the centre manifold. Also, from (1.4.10) $z(t) \approx y(t) - y^2(t)$, which shows that (1.4.6) is approximately true.

The above results are not satisfactory since we have to assume that the initial data is small. In Chapter 2, we show how we can deal with more general initial data. Here we briefly indicate the procedure involved there. If $y_0 \neq -1$, then

$$(y,w,\varepsilon) = (y_0, y_0^2(1+y_0)^{-1}, 0)$$

is a curve of equilibrium points for (1.4.8). Thus, we expect that there is an invariant manifold $w = h(y,\varepsilon)$ for (1.4.8) defined for ε small and $0 \leq y \leq m$, $(m = O(1))$, and with $h(y,\varepsilon)$ close to the curve

$$w = y^2(1+y)^{-1}. \tag{1.4.11}$$

For initial data close to the curve given by (1.4.11), the stability properties of (1.4.8) are the same as the stability properties of the reduced equation

$$\dot{u} = f(u,h(u,\varepsilon)).$$

1.5. Bifurcation Theory

Consider the system of ordinary differential equations

$$\begin{aligned} \dot{w} &= F(w,\varepsilon) \\ F(0,\varepsilon) &\equiv 0 \end{aligned} \tag{1.5.1}$$

where $w \in \mathbb{R}^{n+m}$ and ε is a p-dimensional parameter. We say that $\varepsilon = 0$ is a bifurcation point for (1.5.1) if the qualitative nature of the flow changes at $\varepsilon = 0$, that is, if in any neighborhood of $\varepsilon = 0$ there exist points ε_1 and ε_2 such that the corresponding local phase portraits of $(1.5.1)_{\varepsilon_1}$ and $(1.5.1)_{\varepsilon_2}$ are not topologically equivalent.

Suppose that the linearization of (1.5.1) about $w = 0$ is

$$\dot{w} = C(\varepsilon)w. \qquad (1.5.2)$$

If the eigenvalues of $C(0)$ all have non-zero real parts then, for small $|\varepsilon|$, small solutions of (1.5.1) behave like solutions of (1.5.2) so that $\varepsilon = 0$ is not a bifurcation point. Thus, from the point of view of local bifurcation theory the only interesting situation is when $C(0)$ has eigenvalues with zero real parts.

Suppose that $C(0)$ has n eigenvalues with zero real parts and m eigenvalues whose real parts are negative. We are assuming that $C(0)$ does not have any positive eigenvalues since we are interested in the bifurcation of stable phenomena.

Because of our hypothesis about the eigenvalues of $C(0)$ we can rewrite (1.5.1) as

$$\begin{aligned} \dot{x} &= Ax + f(x,y,\varepsilon) \\ \dot{y} &= By + g(x,y,\varepsilon) \qquad (1.5.3) \\ \dot{\varepsilon} &= 0 \end{aligned}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A is an $n \times n$ matrix whose eigenvalues all have zero real parts, B is an $m \times m$ matrix whose eigenvalues all have negative real parts, and f and g vanish together with each of their derivatives at $(x,y,\varepsilon) = (0,0,0)$.

By Theorem 1, (1.5.3) has a centre manifold $y = h(x,\varepsilon)$, $|x| < \delta_1$, $|\varepsilon| < \delta_2$. By Theorem 2 the behavior of small solutions of (1.5.3) is governed by the equation

$$\dot{u} = Au + f(u,h(u),\varepsilon)$$
$$\dot{\varepsilon} = 0. \tag{1.5.4}$$

In applications n is frequently 1 or 2 so this is a very useful reduction. The reduction to a lower dimensional problem is analogous to the use of the Lyapunov-Schmidt procedure in the analysis of state problems. For the relationship between centre manifold theory and other perturbation techniques such as amplitude expansions, see [12].

1.6. Comments on the Literature

Theorems 1-3 are the simplest such results in centre manifold theory and we briefly mention some of the possible generalizations.

(1) The assumption that the eigenvalues of the linearized problem all have non-positive real parts is not necessary.

(2) The equations need not be autonomous.

(3) In certain circumstances we can replace 'equilibrium point' by 'invariant set'.

(4) Similar results can be obtained for certain classes of infinite-dimensional evolution equations, such as partial differential equations.

There is a vast literature on invariant manifold theory [1,6,18,19,23,24,26,28,30,31,36,37,38,40,42]. For applications of invariant manifold theory to bifurcation theory see [1,12,13,14,15,20,27,30,32,33,34,39,40,41,42,45,50]. For a simple discussion of stable and unstable manifolds see [18, Chapter 13] or [23, Chapter 3].

In Chapter 2 we prove Theorems 1-3. Our proofs of Theorems 1 and 2 are modelled on Kelly [37,38]. Theorem 3 is a special case of a result of Henry [30] and our proof follows his. Throughout Chapter 2 we use methods that generalize to infinite dimensional problems in an obvious way.

CHAPTER 2

PROOFS OF THEOREMS

2.1. Introduction

In this chapter we give proofs of the three main theorems stated in Chapter 1. The proofs are essentially applications of the contraction mapping principle. The procedure used for defining the mappings is rather involved, so we first give a simple example to help clarify the technique. The proofs that we give can easily be extended to the corresponding infinite dimensional case; indeed essentially all we have to do is to replace the norm $|\cdot|$ in finite dimensional space by the norm $||\cdot||$ in a Banach space.

2.2. A Simple Example

We consider a simple example to illustrate the method that we use to prove the existence of centre manifolds.

Consider the system

$$\dot{x}_1 = \dot{x}_2, \ \dot{x}_2 = 0, \quad \dot{y} = -y + g(x_1,x_2), \qquad (2.2.1)$$

where g is smooth and $g(x_1,x_2) = O(x_1^2+x_2^2)$ as $(x_1,x_2) \to (0,0)$. We prove that (2.2.1) has a local centre manifold.

Let $\psi: \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with compact support such that $\psi(x_1,x_2) = 1$ for (x_1,x_2) in a neighborhood of the origin. Define G by $G(x_1,x_2) = \psi(x_1,x_2)g(x_1,x_2)$. We prove that the system of equations

$$\dot{x}_1 = x_2,\ \dot{x}_2 = 0,\ \dot{y} = -y + G(x_1,x_2), \tag{2.2.2}$$

has a centre manifold $y = h(x_1,x_2)$, $(x_1,x_2) \in \mathbb{R}^2$. Since $G(x_1,x_2) = g(x_1,x_2)$ in a neighborhood of the origin, this proves that $y = h(x_1,x_2)$, $x_1^2 + x_2^2 < \delta$ for some δ, is a local centre manifold for (2.2.1).

The solution of the first two equations in (2.2.2) is $x_1(t) = z_1 + z_2t$, $x_2(t) = z_2$, where $x_i(0) = z_i$. If $y(t) = h(x_1(t),x_2(t))$ is a solution of the third equation in (2.2.2) then

$$\frac{d}{dt} h(z_1+z_2t,z_2) = -h(z_1+z_2t,z_2) + G(z_1+z_2t,z_2). \tag{2.2.3}$$

To determine a centre manifold for (2.2.2) we must single out a special solution of (2.2.3). Since $G(x_1,x_2)$ is small for all x_1 and x_2, solutions of the third equation in (2.2.2) behave like solutions of the linearized equation $\dot{y} = -y$. The general solution of (2.2.2) therefore contains a term like e^{-t}. As $t \to \infty$, this component approaches the origin perpendicular to the z_1,z_2 plane. Since the centre manifold is tangent to the z_1,z_2 plane at the origin we must eliminate the e^{-t} component, that is we must eliminate the component that approaches the origin along the stable manifold as $t \to \infty$. To do this we solve (2.2.3) together with the condition

$$\lim_{t\to-\infty} h(z_1+z_2t,z_2)e^t = 0. \tag{2.2.4}$$

Integrating (2.2.3) between $-\infty$ and 0 and using (2.2.4) we obtain

$$h(z_1,z_2) = \int_{-\infty}^{0} e^{s} G(z_1+z_2 s, z_2)\,ds.$$

By construction, $y = h(z_1,z_2)$ is an invariant manifold for (2.2.2). Using the fact that G has compact support and that $G(x_1,x_2)$ has a second order zero at the origin it follows that $h(z_1,z_2)$ has a second order zero at the origin, that is h is a centre manifold.

2.3. Existence of Centre Manifolds

In this section we prove that the system

$$\begin{aligned} \dot{x} &= Ax + f(x,y) \\ \dot{y} &= By + g(x,y) \end{aligned} \qquad (2.3.1)$$

has a centre manifold. As before $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, the eigenvalues of A have zero real parts, the eigenvalues of B have negative real parts and f and g are C^2 functions which vanish together with their derivatives at the origin.

Theorem 1. Equation (2.3.1) has a local centre manifold $y = h(x)$, $|x| < \delta$, where h is C^2.

Proof. As in the example given in the previous section, we prove the existence of a centre manifold for a modified equation. Let $\psi : \mathbb{R}^n \to [0,1]$ be a C^∞ function with $\psi(x) = 1$ when $|x| \le 1$

and $\psi(x) = 0$ when $|x| \geq 2$. For $\varepsilon > 0$ define F and G by

$$F(x,y) = f(x\psi(\tfrac{x}{\varepsilon}),y), \quad G(x,y) = g(x\psi(\tfrac{x}{\varepsilon}),y).$$

The reason that the cut-off function ψ is only a function of x is that the proof of the existence of a centre manifold generalizes in an obvious way to infinite dimensional problems.

We prove that the system

$$\begin{aligned} \dot{x} &= Ax + F(x,y) \\ \dot{y} &= By + G(x,y) \end{aligned} \qquad (2.3.2)$$

has a centre manifold $y = h(x)$, $x \in \mathbb{R}^n$, for small enough ε. Since F and G agree with f and g in a neighborhood of the origin, this proves the existence of a local centre manifold for (2.3.1).

For $p > 0$ let X be the set of Lipschitz functions $h: \mathbb{R}^n \to \mathbb{R}^m$ with Lipschitz constant p, $|h(x)| \leq p$ for $x \in \mathbb{R}^n$ and $h(0) = 0$. With the supremum norm $||\cdot||$, X is a complete space.

For $h \in X$ and $x_0 \in \mathbb{R}^n$, let $x(t,x_0,h)$ be the solution of

$$\dot{x} = Ax + F(x,h(x)), \quad x(0,x_0,h) = x_0. \qquad (2.3.3)$$

The bounds on F and h ensure that the solution of (2.3.3) exists for all t. We now define a new function Th by

$$(Th)(x_0) = \int_{-\infty}^{0} e^{-Bs} G(x(s,x_0,h),h(x(s,x_0,h)))ds. \qquad (2.3.4)$$

If h is a fixed point of (2.3.4) then h is a centre manifold for (2.3.2). We prove that for p and ε small enough, T is a contraction on X.

Using the definitions of F and G, there is a continuous function $k(\varepsilon)$ with $k(0) = 0$ such that

$$\begin{aligned} &|F(x,y)| + |G(x,y)| \leq k(\varepsilon), \\ &|F(x,y) - F(x',y')| \leq k(\varepsilon)[|x-x'| + |y-y'|], \qquad (2.3.5) \\ &|G(x,y) - G(x',y')| \leq k(\varepsilon)[|x-x'| + |y-y'|], \end{aligned}$$

for all $x, x' \in \mathbb{R}^n$ and all $y, y' \in \mathbb{R}^m$ with $|y|, |y'| < \varepsilon$

Since the eigenvalues of B all have negative real parts, there exist positive constants β, C such that for $s \leq 0$ and $y \in \mathbb{R}^m$,

$$|e^{-Bs}y| \leq Ce^{\beta s}|y|. \qquad (2.3.6)$$

Since the eigenvalues of A all have zero real parts, for each $r > 0$ there is a constant $M(r)$ such that for $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$|e^{As}x| \leq M(r)e^{r|s|}|x|. \qquad (2.3.7)$$

Note that in general, $M(r) \to \infty$ as $r \to 0$.

If $p < \varepsilon$, then we can use (2.3.5) to estimate terms involving $G(x(s,x_0,h),h(s,x_0,h))$ and similar terms. We shall suppose that $p < \varepsilon$ from now on.

If $x_0 \in \mathbb{R}^n$, then using (2.3.6) and the estimates on G and h, we have from (2.3.4) that

$$|Th(x_0)| \le C\beta^{-1}k(\varepsilon). \tag{2.3.8}$$

Now let $x_0, x_1 \in \mathbb{R}^n$. Using (2.3.7) and the estimates on F and h, we have from (2.3.3) that for $r > 0$ and $t \le 0$,

$$|x(t,x_0,h) - x(t,x_1,h)| \le M(r)e^{-rt}|x_0-x_1|$$
$$+ 2M(r)k(\varepsilon)\int_t^0 e^{r(s-t)}|x(s,x_0,h) - x(s,x_1,h)|ds.$$

By Gronwall's inequality, for $t \le 0$,

$$|x(t,x_0,h) - x(t,x_1,h)| \le M(r)|x_1-x_0|e^{-\gamma t}, \tag{2.3.9}$$

where $\gamma = r + 2M(r)k(\varepsilon)$. Using (2.3.9) and the bounds on G and h, we obtain from (2.3.4)

$$|Th(x_0) - Th(x_1)| \le C(M(r) + p)k(\varepsilon)(\beta-\gamma)^{-1}|x_0-x_1| \tag{2.3.10}$$

if ε and r are small enough so that $\beta > \gamma$.

Similarly, if $h_1, h_2 \in X$ and $x_0 \in \mathbb{R}^n$, we obtain

$$|Th_1(x_0) - Th_2(x_0)| \le Ck(\varepsilon)r^{-1}(\beta-\gamma)^{-1}[4M(r)k^2(\varepsilon) + 1]\,||h_1-h_2||. \tag{2.3.11}$$

By a suitable choice of p, ε and r, we see from (2.3.8), (2.3.10) and (2.3.11) that T is a contraction on X. This proves the existence of a Lipschitz centre manifold for (2.3.2). To prove that h is C^1 we show that T is a contraction on a subset of X consisting of Lipschitz differentiable functions. The details are similar to the proof given above so we omit the details. To prove that h is C^2 we imitate the proof of Theorem 4.2 on page 333 of [18].

2.4. Reduction Principle

The flow on the centre manifold is governed by the n-dimensional system

$$\dot{u} = Au + f(u,h(u)). \tag{2.4.1}$$

In this section we prove a theorem which enables us to relate the asymptotic behavior of small solutions of (2.3.1) to solutions of (2.4.1).

We first prove a lemma which describes the stability properties of the centre manifold.

Lemma 1. Let $(x(t),y(t))$ be a solution of (2.3.2) with $|(x(0),y(0))|$ sufficiently small. Then there exist positive C_1 and μ such that

$$|y(t) - h(x(t))| \leq C_1 e^{-\mu t} |y(0) - h(x(0))|$$

for all $t \geq 0$.

Proof. Let $(x(t),y(t))$ be a solution of (2.3.2) with $(x(0),y(0))$ sufficiently small. Let $z(t) = y(t) - h(x(t))$, then by an easy computation

$$\dot{z} = Bz + N(x,z) \tag{2.4.2}$$

where

$$N(x,z) = h'(x)[F(x,h(x)) - F(x,z+h(x))] + G(x,z+h(x)) - G(x,h(x)).$$

Using the definitions of F and G and the bounds on h, there is a continuous function $\delta(\varepsilon)$ with $\delta(0) = 0$ such that $|N(x,z)| < \delta(\varepsilon)|z|$ if $|z| < \varepsilon$. Using (2.3.6) we obtain, from (2.4.2),

$$|z(t)| \leq Ce^{-\beta t}|z(0)| + C\delta(\varepsilon)\int_0^t e^{-\beta(t-s)}|z(s)|ds$$

and the result follows from Gronwall's inequality.

Before giving the main result in this section we make some remarks about the matrix A. Since the eigenvalues of A all have zero real parts, by a change of basis we can put A is the form $A = A_1 + A_2$ where A_2 is nilpotent and

$$|e^{A_1 t}x| = |x|. \tag{2.4.3}$$

Since A_2 is nilpotent, we can choose the basis such that

$$|A_2 x| \le (\beta/4)|x|, \qquad (2.4.4)$$

where β is defined by (2.3.6).

We assume for the rest of this section that a basis has been chosen so that (2.4.3) and (2.4.4) hold.

Theorem 2. (a) Suppose that the zero solution of (2.4.1) is stable (asymptotically stable)(unstable). Then the zero solution of (2.3.1) is stable (asymptotically stable)(unstable).

(b) Suppose that the zero solution of (2.3.1) is stable. Let $(x(t),y(t))$ be a solution of (2.3.1) with $(x(0),y(0))$ sufficiently small. Then there exists a solution $u(t)$ of (2.4.1) such that as $t \to \infty$,

$$\begin{aligned} x(t) &= u(t) + O(e^{-\gamma t}) \\ y(t) &= h(u(t)) + O(e^{-\gamma t}) \end{aligned} \qquad (2.4.5)$$

where $\gamma > 0$ is a constant depending only on B.

Proof. If the zero solution of (2.4.1) is unstable then by invariance, the zero solution of (2.3.1) is unstable. From now on we assume that the zero solution of (2.3.1) is stable. We prove that (2.4.5) holds where $(x(t),y(t))$ is a solution of (2.3.2) with $|(x(0),y(0))|$ sufficiently small. Since F and G are equal to f and g in a neighborhood of the origin this proves Theorem 2. We divide the proof into two steps.

I. Let $u_0 \in \mathbb{R}^n$ and $z_0 \in \mathbb{R}^m$ with $|(u_0,z_0)|$ sufficiently small. Let $u(t)$ be the solution of (2.4.1) with $u(0) = u_0$. We prove that there exists a solution $(x(t),y(t))$ of (2.3.2) with $y(0) - h(x(0)) = z_0$ and $x(t) - u(t)$, $y(t) - h(u(t))$ exponentially small as $t \to \infty$.

II. By Step I we can define a mapping S from a neighborhood of the origin in $\mathbb{R}^{n+m}$ into $\mathbb{R}^{n+m}$ by $S(u_0,z_0) = (x_0,z_0)$ where $x_0 = x(0)$. For $|(x_0,z_0)|$ sufficiently small, we prove that (x_0,z_0) is in the range of S.

I. Let $(x(t),y(t))$ be a solution of (2.3.2) and $u(t)$ a solution of (2.4.1). Note that if $u(0)$ is sufficiently small,

$$\dot{u} = Au + F(u,h(u)) \tag{2.4.6}$$

since solutions of (2.4.1) are stable. Let $z(t) = y(t) - h(x(t))$, $\phi(t) = x(t) - u(t)$, then by an easy computation

$$\dot{z} = Bz + N(\phi+u,z) \tag{2.4.7}$$

$$\dot{\phi} = A\phi + R(\phi,z) \tag{2.4.8}$$

where N is defined in the proof of Lemma 1 and

$$R(\phi,z) = F(u+\phi,z+h(u+\phi)) - F(u,h(u)).$$

We now formulate (2.4.7), (2.4.8) as a fixed point problem.

For $a > 0$, $K > 0$, let X be the set of continuous functions $\phi: [0,\infty) \to \mathbb{R}^n$ with $|\phi(t)e^{at}| \leq K$ for all $t \geq 0$. If we define $||\phi|| = \sup\{|\phi(t)e^{at}|: t \geq 0\}$, then X is a complete space. Let u_0, z_0 be sufficiently small and let $u(t)$ be the solution of (2.4.6) with $u(0) = u_0$. Given $\phi \in X$ let $z(t)$ be the solution of (2.4.7) with $z(0) = z_0$. Define $T\phi$ by

$$(T\phi)(t) = -\int_t^\infty e^{A_1(t-s)} [A_2\phi(s) + R(\phi(s),z(s))]ds. \qquad (2.4.9)$$

We solve (2.4.9) by means of the contraction mapping principle. If ϕ is a fixed point of T, then retracing our steps we find that $x(t) = u(t) + \phi(t)$, $y(t) = z(t) + h(x(t))$ is a solution of (2.3.2). We can take a to be as close to β as we please at the cost of increasing K and shrinking the neighborhood on which the result is valid. For simplicity however, we take $K = 1$ and $2a = \beta$ where β is defined by (2.3.6).

Using the bounds on F,G,h and the fact that $N(\phi,0) = 0$, there is a continuous function $k(\varepsilon)$ with $k(0) = 0$ such that if $\phi_1,\phi_2 \in \mathbb{R}^n$ and $z_1,z_2 \in \mathbb{R}^m$ with $|z_i| < \varepsilon$, then

$$|N(\phi_1,z_1) - N(\phi_2,z_2)| \leq k(\varepsilon)[|z_1||\phi_1-\phi_2| + |z_1-z_2|]$$

$$|R(\phi_1,z_1) - R(\phi_2,z_2)| \leq k(\varepsilon)[|z_1-z_2| + |\phi_1-\phi_2|] .$$

From (2.4.7),

$$|z(t)| \leq C|z_0|e^{-\beta t} + Ck(\varepsilon)\int_0^t e^{-\beta(t-s)}|z(s)|ds$$

where we have used (2.3.6) and (2.4.10). By Gronwall's inequality

$$|z(t)| \leq C|z_0|e^{-\beta_1 t} \tag{2.4.11}$$

where $\beta_1 = \beta - Ck(\varepsilon)$. From (2.4.9), if ε is sufficiently small,

$$|T\phi(t)| \leq \frac{e^{-at}}{2} + k(\varepsilon)\int_t^\infty (e^{-as} + C|z_0|e^{-\beta_1 s})ds \leq e^{-at}$$

where we have used (2.4.3), (2.4.4), (2.4.10) and (2.4.11). Hence T maps X into X.

Now let $\phi_1, \phi_2 \in X$ and let z_1, z_2 be the coresponding solutions of (2.4.4) with $z_i(0) = z_0$. We first estimate $w(t) = z_1(t) - z_2(t)$. From (2.4.7),

$$|w(t)| \leq Ck(\varepsilon)\int_0^t e^{-\beta(t-s)}[|z_1(s)||\phi_1(s) - \phi_2(s)| + |w(s)|]ds.$$

Using (2.4.11),

$$|w(t)| \leq C_1 k(\varepsilon)||\phi_1 - \phi_2||e^{-\beta t} + Ck(\varepsilon)\int_0^t e^{-\beta(t-s)}|w(s)|ds$$

where C_1 is a constant, so that by Gronwall's inequality

$$|w(t)| = |z_1(t) - z_2(t)| \leq C_1 k(\varepsilon)||\phi_1 - \phi_2||e^{-\beta_1 t}. \tag{2.4.12}$$

Using (2.4.4) and (2.4.12), for ε sufficiently small,

$$|T\phi_1(t) - T\phi_2(t)| \le \frac{1}{2}\,||\phi_1-\phi_2|| + k(\varepsilon)\int_t^\infty (|\phi_1(s) - \phi_2(s)| + |z_1(s) - z_2(s)|)ds$$

$$< \alpha||\phi_1-\phi_2||$$

where $\alpha < 1$.

The above analysis proves that for each (u_0,z_0) sufficiently small, T has a unique fixed point. If U is a neighborhood of the origin in $\mathbb{R}^{n+m}$ then it is easy to repeat the above analysis to show that $T: X \times U \to X$ is a continuous uniform contraction. This proves that the fixed point depends continuously on u_0 and z_0.

II. Define S by $S(u_0,z_0) = (x_0,z_0)$ where $x_0 = u_0 + \phi(0)$. Since ϕ depends continuously on u_0 and z_0, S is continuous. We prove that S is one-to-one, so that by the Invariance of Domain Theorem (see [9] or [47]) S is an open mapping. Since $S(0,0) = 0$, this proves that the range of S is a full neighborhood of the origin in $\mathbb{R}^{n+m}$.

Proving that S is one-to-one is clearly equivalent to proving that if $u_0 + \phi_0(0) = u_1 + \phi_1(0)$ then $u_0 = u_1$ and $\phi_0(0) = \phi_1(0)$. If $u_0 + \phi_0(0) = u_1 + \phi_1(0)$ then the initial values for x and y are the same, so that by uniqueness of solution of (2.3.2), $u_0(t) + \phi_0(t) = u_1(t) + \phi_1(t)$ for all $t \ge 0$, where $u_i(t)$ is the solution of (2.4.6) with $u_i(0) = u_i$. Hence, for $t \ge 0$,

$$u_0(t) - u_1(t) = \phi_1(t) - \phi_0(t). \tag{2.4.13}$$

Since the real parts of the eigenvalues of A are all zero, $\lim_{t\to\infty} |u_1(t) - u_0(t)|e^{\varepsilon t} = \infty$ for any $\varepsilon > 0$ unless $u_1(0) = u_0(0)$. Also, $|\phi_i(t)| \leq e^{-at}$ for all $t \geq 0$. It now follows from (2.4.13) that S is one-to-one and this completes the proof of the theorem.

2.5. Approximation of the Centre Manifold

For functions $\phi: \mathbb{R}^n \to \mathbb{R}^m$ which are C^1 in a neighborhood of the origin define

$$(M\phi)(x) = \phi'(x)[Ax + f(x,\phi(x))] - B\phi(x) - g(x,\phi(x)).$$

Theorem 3. Suppose that $\phi(0) = 0$, $\phi'(0) = 0$ and that $(M\phi)(x) = O(|x|^q)$ as $x \to 0$ where $q > 1$. Then as $x \to 0$,

$$|h(x) - \phi(x)| = O(|x|^q).$$

Proof. Let $\theta: \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function with compact support such that $\theta(x) = \phi(x)$ for $|x|$ small. Set

$$N(x) = \theta'(x)[Ax + F(x,\theta(x))] - B\theta(x) - G(x,\theta(x)), \qquad (2.5.1)$$

where F and G are defined in Theorem 1. Note that $N(x) = O(|x|^q)$ as $x \to 0$.

In Theorem 1, we proved that h was the fixed point of a contraction mapping $T: X \to X$. Define a mapping S by $Sz = T(z+\theta) - \theta$; the domain of S being a closed subset $Y \subset X$. Since T is a contraction mapping on X, S is a contraction mapping on Y.

For $K > 0$ let

$$Y = \{z \in X: |z(x)| \leq K|x|^q \quad \text{for all} \quad x \in \mathbb{R}^n\}.$$

If we can find a K such that S maps Y into Y then we will have proved the theorem.

We first find an alternative formulation of the map S. For $z \in Y$ let $x(t,x_0)$ be the solution of

$$\dot{x} = Ax + F(x,z(x) + \theta(x)), \quad x(0,x_0) = x(0). \qquad (2.5.2)$$

From (2.3.4)

$$(T(z+\theta))(x_0) = \int_{-\infty}^{0} e^{-Bs} G(x(s,x_0), z(x(s,x_0)) + \theta(x(s,x_0)))ds.$$

Now

$$-\theta(x_0) = -\int_{-\infty}^{0} \frac{d}{ds} [e^{-Bs}\theta(x(s,x_0))]ds$$

$$= \int_{-\infty}^{0} e^{-Bs}[B\theta(x(s,x_0)) - \frac{d}{ds}\theta(x(s,x_0))]ds.$$

Writing x for $x(s,x_0)$ etc., from (2.5.1) and (2.5.2)

$$B\theta(x) - \frac{d}{ds}\theta(x) = B\theta(x) - \theta'(x)[Ax + F(x,z(x) + \theta(x))]$$

$$= -N(x) - G(x,\theta(x)) + \theta'(x)[F(x,\theta) - F(x,z(x) + \theta(x))].$$

Using $Sz = T(z+\theta) - \theta$ and the above calculations

$$(Sz)(x_0) = \int_{-\infty}^{0} e^{-Bs} Q(x(s,x_0),\ z(x(s,x_0)))ds \tag{2.5.3}$$

where $x(s,x_0)$ is the solution of (2.5.2) and

$$Q(x,z) = G(x,\theta+z) - G(x,\theta) - N(x) + \theta'(x)[F(x,\theta) - F(x,\theta+z)] \cdot \tag{2.5.4}$$

We now show that S maps Y into Y for some $K > 0$. By choosing θ suitably, we may assume that $|\theta(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^n$. Since $N(x) = O(|x|^q)$ as $x \to 0$,

$$|N(x)| \leq C_1|x|^q, \quad x \in \mathbb{R}^n \tag{2.5.5}$$

where C_1 is a constant. Now

$$\begin{aligned} |Q(x,z)| &\leq |Q(x,0)| + |Q(x,z) - Q(x,0)| \\ &= |N(x)| + |Q(x,z) - Q(x,0)|. \end{aligned} \tag{2.5.6}$$

We can estimate $|Q(x,z) - Q(x,0)|$ in terms of the Lipschitz constants of F and G. Using (2.3.5), there is a continuous function $k(\varepsilon)$ with $k(0) = 0$, such that

$$|Q(x,z) - Q(x,0)| \leq k(\varepsilon)|z| \tag{2.5.7}$$

for $|z| \leq \varepsilon$. Using (2.5.5), (2.5.6), (2.5.7), for $z \in Y$ and

and $x \in \mathbb{R}^n$, we have that

$$\begin{aligned}|Q(x,z)| &\leq C_1|x|^q + k(\varepsilon)|z(x)| \\ &\leq (C_1+Kk(\varepsilon))|x|^q.\end{aligned} \tag{2.5.8}$$

Using the same calculations as in the proof of Theorem 1, if $x(t,x_0)$ is the solution of (2.5.2), then for each $r > 0$, there is a constant $M(r)$ such that

$$|x(t,x_0)| \leq M(r)|x_0|e^{-\gamma t}, \quad t \leq 0 \tag{2.5.9}$$

where $\gamma = r + 2M(r)k(\varepsilon)$.

Using (2.3.6), (2.5.8) and (2.5.9), if $z \in Y$,

$$|(Sz)(x_0)| \leq C(C_1+Kk(\varepsilon))(M(r))^q(\beta-q\gamma)^{-1}|x_0|^q = C_2|x_0|^q$$

provided ε and r are small enough so that $\beta - q\gamma > 0$. By choosing K large enough and ε small enough, we have that $C_2 \leq K$ and this completes the proof of the theorem.

2.6. Properties of Centre Manifolds

(1) In general (2.3.1) does not have a unique centre manifold. For example, the system $\dot{x} = -x^3$, $\dot{y} = -y$, has the two parameter family of centre manifolds $y = h(x,c_1,c_2)$ where

$$h(x,c_1,c_2)=\begin{cases} c_1 \exp(-\frac{1}{2}x^{-2}), & x > 0 \\ 0 & x = 0 \\ c_2 \exp(-\frac{1}{2}x^{-2}) & x < 0 \end{cases}$$

However, if h and h_1 are two centre manifolds for (2.3.1), then by Theorem 3, $h(x) - h_1(x) = O(|x|^q)$ as $x \to 0$ for all $q > 1$.

(2) If f and g are C^k, $(k \geq 2)$, then h is C^k [37]. If f and g are analytic, then in general (2.3.1) does not have an analytic centre manifold, for example it is easy to show that the system

$$\dot{x} = -x^3, \quad \dot{y} = -y + x^2 \tag{2.6.1}$$

does not have an analytic centre manifold (see exercise (1)).

(3) Centre manifolds need not be unique but there are some points which must always be on any centre manifold. For example, suppose that (x_0,y_0) is a small equilibrium point of (2.3.1) and let $y = h(x)$ be any centre manifold for (2.3.1). Then by Lemma 1 we must have $y_0 = h(x_0)$. Similarly, if Γ is a small periodic orbit of (2.3.1), then Γ must lie on all centre manifolds.

(4) Suppose that $x(t),y(t)$ is a solution of (2.3.1) which *remains in a neighborhood of the origin* for all $t \geq 0$. An examination of the proof of Theorem 2, shows that there is a solution

$u(t)$ of (2.4.1) such that the representation (2.4.5) holds.

(5) In many problems the initial data is not arbitrary, for example, some of the components might always be nonnegative. Suppose $S \subset \mathbb{R}^{n+m}$ with $0 \in S$ and that (2.3.1) defines a local dynamical system on S. It is easy to check, that with the obvious modifications, Theorem 2 is valid when (2.3.1) is studied on S.

Exercise 1. Consider

$$\dot{x} = -x^3, \quad \dot{y} = -y + x^2. \tag{2.6.1}$$

Suppose that (2.6.1) has a centre manifold $y = h(x)$, where h is analytic at $x = 0$. Then

$$h(x) = \sum_{n=2}^{\infty} a_n x^n$$

for small x. Show that $a_{2n+1} = 0$ for all n and that $a_{n+2} = na_n$ for $n = 2,4,\ldots$, with $a_2 = 1$. Deduce that (2.6.1) does not have an analytic centre manifold.

Exercise 2 (Modification of an example due to S.J. van Strien [48]).

If f and g are C^∞ functions, then for each r, (2.3.1) has a C^r centre manifold. However, the size of the neighborhood on which the centre manifold is defined depends on r. The following example shows that in general (2.3.1) does not have a C^∞ centre manifold, even if f and g are analytic.

Consider

$$\dot{x} = -\varepsilon x - x^3, \quad \dot{y} = -y + x^2, \quad \dot{\varepsilon} = 0. \tag{2.6.2}$$

Suppose that (2.6.2) has a C^∞ centre manifold $y = h(x,\varepsilon)$ for $|x| < \delta$, $|\varepsilon| < \delta$. Choose $n > \delta^{-1}$. Then since $h(x,(2n)^{-1})$ is C^∞ in x, there exist constants $a_1, a_2, \ldots, a_{2n}$ such that

$$h(x,(2n)^{-1}) = \sum_{i=1}^{2n} a_i x^i + O(x^{2n+1})$$

for $|x|$ small enough. Show that $a_i = 0$ for odd i and that if $n > 1$,

$$\begin{aligned} &(1 - (2i)(2n)^{-1})a_{2i} = (2i-2)a_{2i-2}, \quad i = 2,\ldots,n \\ &a_2 \neq 0. \end{aligned} \tag{2.6.3}$$

Obtain a contradiction from (2.6.3) and deduce that (2.6.2) does not have a C^∞ centre manifold.

Exercise 3. Suppose that the nonlinearities in (2.3.1) are odd, that is $f(x,y) = -f(-x,-y)$, $g(x,y) = -g(-x,-y)$. Prove that (2.3.1) has a centre manifold $y = h(x)$ with $h(x) = -h(-x)$. [The example $\dot{x} = -x^3$, $\dot{y} = -y$, shows that if h is any centre manifold for (2.3.1) then $h(x) \neq -h(-x)$ in general.]

2.7. Global Invariant Manifolds for Singular Perturbation Problems

To motivate the results in this section we reconsider Example 3 in Chapter 1. In that example we applied centre manifold theory to a system of the form

$$\begin{aligned} y' &= \varepsilon f(y,w) \\ w' &= -w + y^2 - yw + \varepsilon f(y,w) \\ \varepsilon' &= 0 \end{aligned} \qquad (2.7.1)$$

where $f(0,0) = 0$. Because of the local nature of our results on centre manifolds, we only obtained a result concerning small initial data. Let $v = -w(1+y) + y^2$, then we obtain a system of the form

$$\begin{aligned} y' &= \varepsilon g_1(y,v) \\ v' &= -v + \varepsilon g_2(y,v) \\ \varepsilon' &= 0 \end{aligned} \qquad (2.7.2)$$

where $g_i(0,0) = 0$, $i = 1,2$. Note that if $y \neq -1$, then $(y,0,0)$ is always an equilibrium point for (2.7.2) so we expect that (2.7.2) has an invariant manifold $v = h(y,\varepsilon)$ defined for $-1 < y < m$ say and ε sufficiently small.

Theorem 4. Consider the system

$$\begin{aligned} x' &= Ax + \varepsilon f(x,y,\varepsilon) \\ y' &= By + \varepsilon g(x,y,\varepsilon) \\ \varepsilon' &= 0 \end{aligned} \qquad (2.7.3)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and A,B are as in Theorem 1. Suppose also that f,g are C^2 with $f(0,0,0) = 0$, $g(0,0,0) = 0$. Let $m > 0$. Then there is a $\delta > 0$ such that (2.7.3) has an invariant manifold $y = h(x,\varepsilon)$, $|x| < m$, $|\varepsilon| < \delta$, with $|h(x,\varepsilon)| < C|\varepsilon|$, where C is a constant which depends on m,A,B,f and g.

Proof. Let $\psi: \mathbb{R}^n \to [0,1]$ be a C^∞ function with $\psi(x) = 1$ if $|x| \le m$ and $\psi(x) = 0$ if $|x| \ge m + 1$. Define F and G by

$$F(x,y,\varepsilon) = \varepsilon f(x\psi(x),y,\varepsilon), \quad G(x,y,\varepsilon) = \varepsilon g(x\psi(x),y,\varepsilon).$$

We can then prove that the system

$$\begin{aligned} x' &= Ax + F(x,y,\varepsilon) \\ y' &= By + G(x,y,\varepsilon) \end{aligned} \tag{2.7.4}$$

has an invariant manifold $y = h(x,\varepsilon)$, $x \in \mathbb{R}^n$, for $|\varepsilon|$ sufficiently small. The proof is essentially the same as that given in the proof of Theorem 1 so we omit the details.

Remark. If $x = (x_1,x_2,\ldots,x_n)$ then we can similarly prove the existence of $h(x,\varepsilon)$ for $\underline{m}_i < x_i < \overline{m}_i$.

The flow on the invariant manifold is given by the equation

$$u' = Au + \varepsilon f(u,h(u,\varepsilon)). \tag{2.7.5}$$

With the obvious modifications it is easy to show that the stability of solutions of (2.7.3) is determined by equation (2.7.5) and that the representation of solutions given in (2.4.5) holds.

Finally, we state an approximation result.

Theorem 5. Let $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^m$ satisfy $\phi(0,0) = 0$ and $|(M\phi)(x,\varepsilon)| \leq C\varepsilon^p$ for $|x| \leq m$ where p is a positive integer, C is a constant and

$$(M\phi)(x,\varepsilon) = D_x\phi(x,\varepsilon)[Ax + \varepsilon f(x,\phi(x,\varepsilon))] - B\phi(x,\varepsilon) - \varepsilon g(x,\phi(x,\varepsilon)).$$

Then, for $|x| \leq m$,

$$|h(x,\varepsilon) - \varphi(x,\varepsilon)| \leq C_1\varepsilon^p$$

for some constant C_1.

Theorem 5 is proved in exactly the same way as Theorem 3 so we omit the proof.

For further information on the application of centre manifold theory to singular perturbation problems see Fenichel [20] and Henry [30].

CHAPTER 3

EXAMPLES

3.1. Rate of Decay Estimates in Critical Cases

In this section we study the decay to zero of solutions of the equation

$$\ddot{r} + \dot{r} + f(r) = 0 \tag{3.1.1}$$

where f is a smooth function with

$$f(r) = r^3 + ar^5 + O(r^7) \quad \text{as} \quad r \to 0, \tag{3.1.2}$$

where a is a constant. By using a suitable Liapunov function it is easy to show that the zero solution of (3.1.1) is asymptotically stable. However, because $f'(0) = 0$, the rate of decay cannot be determined by linearization.

In [8] the rate of decay of solutions was given using techniques which were special to second order equations. We show how centre manifolds can be used to obtain similar results.

We first put (3.1.1) into canonical form. Let $x = r + \dot{r}$, $y = \dot{r}$, then

$$\begin{aligned} \dot{x} &= -f(x-y) \\ \dot{y} &= -y - f(x-y). \end{aligned} \tag{3.1.3}$$

By Theorem 1 of Chapter 2, (3.1.3) has a centre manifold $y = h(x)$.

By Theorem 2 of Chapter 2, the equation which determines the asymptotic behavior of small solutions of (3.1.3) is

$$\dot{u} = -f(u-h(u)). \tag{3.1.4}$$

Using (3.1.2) and $h(u) = O(u^2)$,

$$\dot{u} = -u^3 + O(u^4). \tag{3.1.5}$$

Without loss of generality we can suppose that the solution $u(t)$ of (3.1.5) is positive for all $t \geq 0$. Using L'Hospital's rule,

$$-1 = \lim_{t\to\infty} \frac{\dot{u}}{u^3} = \lim_{t\to\infty} t^{-1} \int_1^{u(t)} s^{-3} ds.$$

Hence, if $w(t)$ is the solution of

$$\dot{w} = -w^3, \quad w(0) = 1, \tag{3.1.6}$$

then $u(t) = w(t+o(t))$. Since

$$w(t) = \frac{1}{\sqrt{2}} t^{-1/2} + Ct^{-3/2} + O(t^{-5/2}) \tag{3.1.7}$$

where C is a constant, we have that

$$u(t) = \frac{1}{\sqrt{2}} t^{-1/2} + o(t^{-1/2}). \tag{3.1.8}$$

To obtain further terms in the asymptotic expansion of $u(t)$, we need an approximation to $h(u)$. To do this, set

$$(M\phi)(x) = -\phi'(x)f(x-\phi(x)) + \phi(x) + f(x-\phi(x)).$$

If $\phi(x) = -x^3$ then $(M\phi)(x) = O(x^5)$ so that by Theorem 3 of Chapter 2, $h(x) = -x^3 + O(x^5)$. Subsituting this into (3.1.4) we obtain

$$\dot{u} = -u^3 - (a+3)u^5 + O(u^7). \tag{3.1.9}$$

Choose T so that $u(T) = 1$. Dividing (3.1.9) by u^3, integrating over $[T,t]$ and using (3.1.8), we obtain

$$w^{-1}(u(t)) = t + \text{constant} + (3+a)\int_T^t u^2(s)ds \tag{3.1.10}$$

where w is the solution of (3.1.6). Using (3.1.8) and (3.1.9),

$$\begin{aligned}\int_T^t u^2(s)ds &= -\int_0^t \frac{\dot{u}(s)ds}{u(s)} + \int_T^t O(u^4(s))ds \\ &= -\ln t^{-1/2} + \text{constant} + o(1).\end{aligned} \tag{3.1.11}$$

Substituting (3.1.11) into (3.1.10) and using (3.1.7),

$$u(t) = \frac{1}{\sqrt{2}} t^{-1/2} - \frac{t^{-3/2}}{4\sqrt{2}} [(a+3) \ln t + C] + o(t^{-3/2}) \tag{3.1.12}$$

where C is a constant.

If $x(t),y(t)$ is a solution of (3.1.3), it follows from Theorem 2 of Chapter 2 that either $x(t),y(t)$ tend to zero exponentially fast or

$$x(t) = \pm u(t), \ y(t) = \mp u^3(t)$$

where $u(t)$ is given by (3.1.12).

3.2. Hopf Bifurcation

There is an extensive literature on Hopf Bifurcation [1,13,15,16,27,30,35,39,40,42,46], so we only given an outline of the theory. Our treatment is based on [15].

Consider the one-parameter family of ordinary differential equations on $\mathbb{R}^2$,

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^2,$$

such that $f(0,\alpha) = 0$ for all sufficiently small α. Assume that the linearized equation about $z = 0$ has eigenvalues $\gamma(\alpha) \pm i\omega(\alpha)$ where $\gamma(0) = 0$, $\omega(0) = \omega_0 \neq 0$. We also assume that the eigenvalues cross the imaginary axis with nonzero speed so that $\gamma'(0) \neq 0$. Since $\gamma'(0) \neq 0$, by the implicit function theorem we can assume without loss of generality that $\gamma(\alpha) = \alpha$. By means of a change of basis the differential equation takes the form

$$\dot{x} = A(\alpha)x + F(x,\alpha), \tag{3.2.1}$$

where

$$A(\alpha) = \begin{bmatrix} \alpha & -\omega(\alpha) \\ \omega(\alpha) & \alpha \end{bmatrix}$$

$$F(x,\alpha) = O(|x|^2).$$

Under the above conditions, there are periodic solutions of (3.2.1) bifurcating from the zero solution. More precisely, for α small there exists a unique one parameter family of small amplitude periodic solutions of (3.2.1) in exactly one of the cases (i) $\alpha < 0$, (ii) $\alpha = 0$, (iii) $\alpha > 0$. However, further conditions on the nonlinear terms are required to determine the specific type of bifurcation.

Exercise 1. Use polar co-ordinates for

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 - \omega x_2 + K x_1 (x_1^2 + x_2^2) \\ x_2 &= \omega x_1 + \alpha x_2 + K x_2 (x_1^2 + x_2^2) \end{aligned}$$

to show that case (i) applies if $K > 0$ and case (iii) applies if $K < 0$.

To find periodic solutions of (3.2.1) we make the substitution

$$x_1 = \varepsilon r \cos \theta, \quad x_2 = \varepsilon r \sin \theta, \quad \alpha \to \varepsilon \alpha, \qquad (3.2.2)$$

where ε is a function of α. After substituting (3.2.2) into (3.2.1) we obtain a system of the form

$$\begin{aligned} \dot{r} &= \varepsilon[\alpha r + r^2 C_3(\theta, \alpha\varepsilon)] + \varepsilon^2 r^3 C_4(\theta, \alpha\varepsilon) + O(\varepsilon^3) \\ \dot{\theta} &= \omega_0 + O(\varepsilon). \end{aligned} \qquad (3.2.3)$$

We now look for periodic solutions of (3.2.3) with $\varepsilon \to 0$ and r

near a constant r_0. If C_3 and C_4 are independent of θ and the higher order terms are zero then the first equation in (3.2.3) takes the form

$$\dot{r} = \varepsilon[\alpha r + \beta r^2] + \varepsilon^2 K r^3 . \qquad (3.2.4)$$

Periodic solutions are then the circles $r = r_0$, where r_0 is a zero of the right hand side of (3.2.4). We reduce the first equation in (3.2.3) to the form (3.2.4) modulo higher order terms by means of a certain transformation. It turns out that the constant β is zero. Under the hypothesis K is non-zero, it is straightforward to prove the existence of periodic solutions by means of the implicit function theorem. The specific type of bifurcation depends on the sign of K so it is necessary to obtain a formula for K.

Let $F(x,\alpha) = [F_1(x_1,x_2,\alpha),\ F_2(x_1,x_2,\alpha)]^T$ and let

$$F_j(x_1,x_2,\alpha) = B_2^j(x_1,x_2,\alpha) + B_3^j(x_1,x_2,\alpha) + O(x_1^4+x_2^4) \qquad (3.2.5)$$

where B_i^j is a homogeneous polynomial of degree i in (x_1,x_2). Substituting (3.2.2) into (3.2.1) and using (3.2.5) we obtain (3.2.3) where for $i = 3,4$,

$$\begin{aligned} C_i(\theta,\alpha) &= (\cos\theta)B_{i-1}^1(\cos\theta,\ \sin\theta,\alpha) \\ &\quad + (\sin\theta)B_{i-1}^2(\cos\theta,\ \sin\theta,\alpha). \end{aligned} \qquad (3.2.6)$$

Lemma 1. There exists a coordinate change

$$\bar{r} = r + \varepsilon u_1(r,\theta,\alpha,\varepsilon) + \varepsilon^2 u_2(r,\theta,\alpha,\varepsilon)$$

which transforms (3.2.3) into the system

$$\begin{aligned} \dot{\bar{r}} &= \varepsilon\alpha\bar{r} + \varepsilon^2\bar{r}^3 K + O(\varepsilon^3) \\ \dot{\theta} &= \omega_0 + O(\varepsilon) \end{aligned} \qquad (3.2.7)$$

where the constant K is given by

$$K = (1/2\pi)\int_0^{2\pi} [C_4(\theta,0) - \omega_0^{-1} C_3(\theta,0)D_3(\theta,0)]\,d\theta \qquad (3.2.8)$$

where C_3 and C_4 are given by (3.2.6) and

$$D_3(\theta,0) = \cos\theta B_2^2(\cos\theta, \sin\theta,0) - (\sin\theta)B_2^1(\cos\theta, \sin\theta,0).$$

The coordinate change is constructed via averaging. We refer to [15] for a proof of the Lemma.

If $K = 0$ then we must make further coordinate changes. We assume that $K \neq 0$ from now on.

Recall that we are looking for periodic solutions of (3.2.7) with $\varepsilon \to 0$ and $\bar{r}$ near a constant r_0. This suggests that we set $\alpha = -\text{sgn}(K)\varepsilon$ and $r_0 = |K|^{-1/2}$. The next result gives the existence of periodic solutions of

$$\begin{aligned} \dot{\bar{r}} &= \varepsilon^2 [-\text{sgn}(K)\bar{r} + \bar{r}^3 K] + O(\varepsilon^3) \\ \dot{\theta} &= \omega_0 + O(\varepsilon) \end{aligned} \qquad (3.2.9)$$

with $\overline{r} - r_0$ small.

Lemma 2. Equation (3.2.9) has a unique periodic solution for ε small and $\overline{r}$ in a compact region either for $\varepsilon > 0$ (when $K < 0$) or $\varepsilon < 0$ (when $K > 0$). Also

$$\overline{r} = r_0 + O(\varepsilon), \ \theta(t,\varepsilon) = \omega_0 t + O(\varepsilon)$$

and the period of the solution $\tau(\varepsilon)$ is given by

$$\tau(\varepsilon) = (2\pi/\omega_0) + O(\varepsilon).$$

The periodic *solution is stable if* $K < 0$ *and* unstable *if* $K > 0$.

Lemma 2 is proved by a simple application of the implicit function theorem. We again refer to [15] for a proof.

Lemma 2 also proves the existence of a one parameter family of periodic solutions of (3.2.1). We cannot immediately assert that this family is unique however, since we may have lost some periodic solutions by the choice of scaling, i.e. by scaling $\alpha \to \alpha\varepsilon$ and by choosing $\varepsilon = -\mathrm{sgn}(K)\alpha$.

In order to justify the scaling suppose that $x_1 = R\cos\theta$, $x_2 = R\sin\theta$ is a periodic solution of (3.2.1) bifurcating from $x_1 = x_2 = 0$. Then R satisfies

$$\dot{R} = \alpha R + O(R^2).$$

When R attains its maximum, $\dot{R} = 0$ so that $R = O(\alpha)$. This justifies the scaling $\alpha \to \alpha\varepsilon$. A similar argument applied to periodic solutions of (3.2.7) justifies the choice of ε.

Theorem. Suppose that the constant K defined by (3.2.8) is non-zero. Then (3.2.1) has a unique periodic solution bifurcating from the origin, either for $\alpha > 0$ (when $K < 0$) or $\alpha < 0$ (when $K > 0$). If $x = R\cos\theta$, $y = R\sin\theta$ then the periodic solution has the form

$$R(t,\alpha) = |\alpha K^{-1}|^{1/2} + O(|\alpha|)$$

$$\theta(t,\alpha) = \omega_0 t + O(|\alpha|^{1/2})$$

with period $\tau(\alpha) = (2\pi/\omega_0) + O(|\alpha|^{1/2})$. The periodic solution is stable if $K < 0$ and unstable if $K > 0$.

Finally, we note that since the value of K depends only on the nonlinear terms evaluated at $\alpha = 0$, when applying the above Theorem to (3.2.1) we only need assume that the eigenvalues of $A(\alpha)$ cross the imaginary axis with non-zero speed and that

$$A(0) = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}$$

with ω_0 non-zero.

3.3. Hopf Bifurcation in a Singular Perturbation Problem

In this section we study a singular perturbation problem which arises from a mathematical model of the immune response to antigen [43]. The equations are

$$
\begin{aligned}
\varepsilon\dot{x} &= -[x^3 + (a - \tfrac{1}{2})x + b - \tfrac{1}{2}] \\
\dot{a} &= \tfrac{1}{2}\,\delta(1-x) - a - \gamma_1 ab \qquad (3.3.1) \\
\dot{b} &= -\gamma_1 ab + \gamma_2 b
\end{aligned}
$$

where $\varepsilon, \delta, \gamma_1, \gamma_2$ are positive parameters. In the above model a and b represent certain concentrations so they must be non-negative. Also, x measures the stimulation of the system and it is scaled so that $|x| \leq 1$. The stimulation is assumed to take place on a much faster time scale than the response so that ε is very small.

The above problem was studied in [43] and we briefly outline the method used by Merrill to prove the existence of periodic solutions of (3.3.1). Putting $\varepsilon = 0$ in the first equation in (3.3.1) we obtain

$$x^3 + (a - \tfrac{1}{2})x + b - \tfrac{1}{2} = 0. \qquad (3.3.2)$$

Solving (3.3.2) for x as a function of a and b we obtain $x = F(a,b)$ and substituting this into the second equation in (3.3.1) we obtain

$$
\begin{aligned}
\dot{a} &= \tfrac{1}{2}\,\delta(1-F(a,b)) - a - \gamma_1 ab \\
\dot{b} &= -\gamma_1 ab + \gamma_2 b.
\end{aligned}
\qquad (3.3.3)
$$

Using δ as the parameter, it was shown that relative to a certain equilibrium point (a_0, b_0) a Hopf bifurcation takes place in (3.3.3). By appealing to a result in singular perturbation theory, it was concluded that for ε sufficiently small, (3.3.1) also has a periodic solution.

We use the theory given in Chapter 2 to obtain a similar result. Let (x_0, a_0, b_0) be a fixed point of (3.3.1). If $b_0 \neq 0$ then $a_0 = \gamma_2/\gamma_1$ and x_0, a_0 satisfy

$$\begin{aligned} & x_0^3 + \left(\frac{\gamma_2}{\gamma_1} - \frac{1}{2}\right)x_0 + b_0 - \frac{1}{2} = 0, \\ & \frac{1}{2}\,\delta(1-x_0) - \frac{\gamma_2}{\gamma_1} - \gamma_2 b_0 = 0. \end{aligned} \qquad (3.3.4)$$

Recall also that for the biological problem, we must have $b_0 \geq 0$ and $|x_0| \leq 1$. We assume for the moment that x_0 and b_0 satisfy (3.3.4) and these restrictions. The reality of these solutions are considered later. We let $a_0 = \gamma_2/\gamma_1$ for the rest of this section.

Let $y = a - a_0$, $z = b - b_0$, $w = -\psi(x-x_0) - x_0 y - z$ where $\psi = 3x_0^2 + a_0 - \frac{1}{2}$. Then assuming ψ is non-zero,

$$\begin{aligned} \varepsilon\dot{w} &= g(w,y,z,\varepsilon) \\ \dot{y} &= f_2(w,y,z,\varepsilon) \\ \dot{z} &= f_3(w,y,z,\varepsilon) \end{aligned} \qquad (3.3.5)$$

where

$$\begin{aligned}
g(w,y,z,\varepsilon) &= f_1(w,y,z,\varepsilon) - \varepsilon x_0 f_2(w,y,z,\varepsilon) - \varepsilon f_3(w,y,z,\varepsilon) \\
f_1(w,y,z,\varepsilon) &= -\psi w + N(w+x_0y+z,y) \\
f_2(w,y,z,\varepsilon) &= (\tfrac{\delta}{2}\psi^{-1}x_0 - 1 - \gamma_1 b_0)y + (\tfrac{\delta}{2}\psi^{-1} - \gamma_2)z + \tfrac{\delta}{2}\psi^{-1}w - \gamma_1 yz \\
f_3(w,y,z,\varepsilon) &= -\gamma_1 b_0 y - \gamma_1 yz \\
N(\theta,y) &= -\psi^{-2}\theta^3 + 3\psi^{-1}x_0\theta^2 - y\theta .
\end{aligned}$$

In order to apply centre manifold theory we change the time scale by setting $t = \varepsilon s$. We denote differentiation with respect to s by $'$ and differentiation with respect to t by $\cdot$. Equation (3.3.5) can now be written in the form

$$\begin{aligned}
w' &= g(w,y,z,\varepsilon) \\
y' &= \varepsilon f_2(w,y,z,\varepsilon) \\
z' &= \varepsilon f_3(w,y,z,\varepsilon) \\
\varepsilon' &= 0 .
\end{aligned} \qquad (3.3.6)$$

Suppose that $\psi > 0$. Then the linearized system corresponding to (3.3.6) has one negative eigenvalue and three zero eigenvalues. By Theorem 1 of Chapter 2, (3.3.6) has a centre manifold $w = h(y,z,\varepsilon)$. By Theorem 2 of Chapter 2, the local behavior of solutions of (3.3.6) is determined by the equation

$$\begin{aligned}
y' &= \varepsilon f_2(h(y,z,\varepsilon),y,z,\varepsilon) \\
z' &= \varepsilon f_3(h(y,z,\varepsilon),y,z,\varepsilon)
\end{aligned} \qquad (3.3.7)$$

or in terms of the original time scale

$$\begin{aligned}\dot{y} &= f_2(h(y,z,\varepsilon),y,z,\varepsilon)\\ \dot{z} &= f_3(h(y,z,\varepsilon),y,z,\varepsilon).\end{aligned} \tag{3.3.8}$$

We now apply the theory given in the previous section to show that (3.3.8) has a periodic solution bifurcating from the origin for certain values of the parameters.

The linearization of the vector field in (3.3.8) about $y = z = 0$ is given by

$$J(\varepsilon) = \begin{bmatrix} \frac{\delta}{2}\psi^{-1}x_0 - 1 - \gamma_1 b_0 & \frac{\delta}{2}\psi^{-1} - \gamma_2 \\ -\gamma_1 b_0 & 0 \end{bmatrix} + O(\varepsilon).$$

If (3.3.8) is to have a Hopf bifurcation then we must have $\mathrm{trace}(J(\varepsilon)) = 0$ and $\frac{\delta}{2}\psi^{-1} - \gamma_2 > 0$. From the previous analysis, we must also have that x_0, b_0 are solutions of (3.3.4) with $|x_0| < 1$, $b_0 > 0$ and $\psi > 0$. We do not attempt to obtain the general conditions under which the above conditions are satisfied, we only work out a special case.

Lemma. Let $\gamma_1 < 2\gamma_2$. Then for each $\varepsilon > 0$, there exists $\delta(\varepsilon)$, $x_0(\varepsilon)$, $b_0(\varepsilon)$ such that $0 < 2x_0(\varepsilon) < 1$, $b_0(\varepsilon) > 0$, $\psi > 0$, $\delta(\varepsilon)\psi^{-1} - 2\gamma_2 > 0$, trace $J(\varepsilon) = 0$ and (3.3.4) is satisfied.

Proof. Fix γ_1 and γ_2 with $\gamma_1 < 2\gamma_2$. If x_0, b_0, δ satisfy the second equation in (3.3.4) then

$$\operatorname{trace}(J(0)) = \frac{\delta}{2}\left[x_0\psi^{-1} - \frac{\gamma_1}{\gamma_2}(1-x_0)\right]$$

It is easy to show that there is a unique $x_0(0) \in (0,\frac{1}{2})$ that satisfies $\operatorname{trace}(J(0)) = 0$. Clearly $\psi > 0$ for this choice of $x_0(0)$. We now obtain $b_0(0)$ and $\delta(0)$ as the unique solution of (3.3.4) and an easy computation shows that $b_0(0) > 0$, $\delta(0) > 0$ and $\delta(0)\psi^{-1} - 2\gamma_2 > 0$. By the implicit function theorem, for ε, $x_0 - x_0(0)$, $b_0 - b_0(0)$ sufficiently small, there exists $\delta(\varepsilon, x_0, b_0) = \delta(0) + O(\varepsilon)$ such that

$$\operatorname{trace}(J(\varepsilon)) = \frac{\delta}{2}\psi^{-1}x_0 - 1 - \gamma_1 b_0 + O(\varepsilon) = 0.$$

After substituting $\delta = \delta(\varepsilon, x_0, b_0)$ into (3.3.4), another application of the implicit function theorem gives the result. This completes the proof of the Lemma.

From now on we fix γ_1 and γ_2 with $\gamma_1 < 2\gamma_2$. Using the same calculations as in the Lemma, for each ε and δ with ε and $\delta - \delta(\varepsilon)$ sufficiently small there is a solution $x_0(\varepsilon,\delta)$, $b_0(\varepsilon,\delta)$ of (3.3.4). Writing $x_0 = x_0(\varepsilon,\delta(\varepsilon))$ and trace(J) as a function of δ, we have that

$$\frac{\partial}{\partial\delta}(\operatorname{trace}(J(\delta)))\Big|_{\delta=\delta(\varepsilon)} = \frac{\gamma_1\delta(\varepsilon)\psi^{-1}}{4(1-x_0)}\left[6\gamma_1 x_0^3 + \right.$$

$$\left. + (2\gamma_2-\gamma_1)x_0 - 12\gamma_1 x_0\right] + O(\varepsilon) < 0,$$

if ε is sufficiently small. Hence, the eigenvalues of $J(\delta)$ cross the imaginary axis with non-zero speed at $\delta = \delta(\varepsilon)$.

Let $y_1 = \ell(\varepsilon)z$, $z_1 = m(\varepsilon)y$ where

$$\ell(\varepsilon) = (\gamma_1 b_0(\varepsilon))^{-1/2} + O(\varepsilon), \quad m(\varepsilon) = [(\delta/2)\psi^{-1} - \gamma_2]^{1/2} + O(\varepsilon).$$

For $\delta = \delta(\varepsilon)$, (3.3.8) in these new coordinates becomes

$$\dot{y}_1 = -\omega_0 z_1 - \gamma_1 m^{-1}(\varepsilon) y_1 z_1$$
$$\dot{z}_1 = \omega_0 y_1 + m(\varepsilon)(\delta/2)\psi^{-1} h(m^{-1}(\varepsilon)z_1, \ell^{-1}(\varepsilon)y_1, \varepsilon) - \gamma_1 \ell^{-1}(\varepsilon) y_1 z_1$$

where

$$\omega_0^2(\varepsilon) = \gamma_1 b_0(\varepsilon)[(\delta/2)\psi^{-1} - \gamma_2] + O(\varepsilon).$$

To apply the results of the last section, we need to calculate the $K(\varepsilon)$ associated with (3.3.9). We shall show how $K(0)$ can be calculated; if $K(0)$ is non-zero then $K(\varepsilon)$ will be non-zero also. To calculate $K(0)$ we need to know the quadratic and cubic terms in (3.3.9) when $\varepsilon = 0$. Thus, we have to find $h(y,z,0)$ modulo fourth order terms.

Let

$$(M\phi)(y,z) = -g(\phi(y,z),y,z,0) \qquad (3.3.10)$$

Then by Theorem 2 of Chapter 2, if we can find ϕ such that

$(M\phi)(y,z) = O(y^4+z^4)$ then $h(y,z,0) = \phi(y,z) + O(y^4+z^4)$. Suppose that

$$\phi = \phi_2 + \phi_3 \qquad (3.3.11)$$

where ϕ_j is a homogeneous polynomial of degree j. Substituting (3.3.11) into (3.3.10) we obtain

$$(M\phi)(y,z) = \psi\phi_2(y,z) - 3\psi^{-1}x_0(x_0y+z)^2 + y(x_0y+z) + O(|y|^3 + |z|^3).$$

Hence, if

$$\phi_2(y,z) = 3\psi^{-2}x_0(x_0y+z)^2 - \psi^{-1}y(x_0y+z) \qquad (3.3.12)$$

then $(M\phi)(y,z) = O(|y|^3 + |z|^3)$. Substituting (3.3.11) into (3.3.10) with $\phi_2(y,z)$ given by (3.3.12), we obtain

$$(M\phi)(y,z) = \psi\phi_3(y,z) + \psi^{-2}(x_0y+z)^3 - 6\psi^{-1}x_0(x_0y+z)\phi_2(y,z)$$
$$+ y\phi_2(y,z) + O(y^4+z^4).$$

Hence,

$$h(y,z) = \phi_2(y,z) - \psi^{-3}(x_0y+z)^3 + 6\psi^{-2}x_0(x_0y+z)\phi_2(y,z) -$$
$$- \psi^{-1}y\phi_2(y,z) + O(y^4+z^4)$$

where $\phi_2(y,z)$ is defined by (3.3.12). K(0) can now be calculated as in the previous section. The sign of $K(0)$ will depend on γ_1 and γ_2. If $K(0)$ is non-zero then we can apply Theorem 1 of Section 2 to prove the existence of periodic solutions of (3.3.1). The stability of the periodic solutions is determined by the sign of $K(0)$. If $K(0)$ is zero then we have to calculate $K(\varepsilon)$.

CHAPTER 4

BIFURCATIONS WITH TWO PARAMETERS IN TWO DIMENSIONS

4.1. Introduction

In this chapter we consider an autonomous ordinary differential equation in the plane depending on a two-dimensional parameter ε. We suppose that the origin $x = 0$ is a fixed point for all ε. More precisely, we consider

$$\begin{gathered} \dot{x} = f(x,\varepsilon), \quad x \in \mathbb{R}^2, \ \varepsilon = (\varepsilon_1,\varepsilon_2) \in \mathbb{R}^2, \\ f(0,\varepsilon) \equiv 0. \end{gathered} \tag{4.1.1}$$

The linearized equation about $x = 0$ is

$$\dot{x} = A(\varepsilon)x,$$

and we suppose that $A(0)$ has two zero eigenvalues. The object is to study small solutions of (4.1.1) for $(\varepsilon_1,\varepsilon_2)$ in a full neighborhood of the origin. More specifically, we wish to divide a neighborhood of $\varepsilon = 0$ into distinct components, such that if $\varepsilon,\bar{\varepsilon}$ are in the same component, then the phase portraits of $(4.1.1)_\varepsilon$ and $(4.1.1)_{\bar{\varepsilon}}$ are topologically equivalent. We also want to describe the behavior of solutions for each component. The boundaries of the components correspond to bifurcation points.

Since the eigenvalues of $A(0)$ are both zero we have that either (i) $A(0)$ is the zero matrix, or (ii) $A(0)$ has a Jordan block,

$$A(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} .$$

There is a distinction between (i) and (ii) even for the study of fixed points. Under generic assumptions, in Case (ii), equation (4.1.1) has exactly 2 fixed points in a neighborhood of the origin. For Case (i) the situation is much more complicated [17, 25].

Another distinction arises when we consider the eigenvalues of $A(\varepsilon)$. We would expect the nature of the eigenvalues of $A(\varepsilon)$ to determine (in part) the possible types of bifurcation. If

$$A(\varepsilon) = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}$$

then the eigenvalues of $A(\varepsilon)$ are always real so we do not expect to obtain periodic orbits surrounding the origin. On the other hand if

$$A(\varepsilon) = \begin{bmatrix} 0 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{bmatrix} \tag{4.1.2}$$

then the range of the eigenvalues of $A(\varepsilon)$ is a neighborhood of the origin in $\mathbb{C}$, that is, if z is a small complex number then $A(\varepsilon)$ has an eigenvalue z for some ε.

We shall assume from now on that $A(\varepsilon)$ is given by (4.1.2.).

Takens [49, 50] and Bodganov [see 2] have studied normal forms for local singularities. Takens shows, for example, that any perturbation of the equation

$$\dot{x}_1 = x_2 + x_1^2, \quad \dot{x}_2 = -x_1^2$$

is topologically equivalent to

$$\dot{x}_1 = x_2 + x_1^2, \quad \dot{x}_2 = \varepsilon_1 + \varepsilon_2 x_1 - x_1^2$$

for some $\varepsilon_1, \varepsilon_2$. *There are certain difficulties in applying these* results since we must transform our equation into normal form, modulo higher order terms.

In [39, p. 333-348], Kopell and Howard study (4.1.1) under the assumptions that $A(\varepsilon)$ is given by (4.1.2) and that

$$\frac{\partial^2 f_2}{\partial x_1^2}(0,0) \neq 0$$

where f_2 is the second component of f. Their approach consists of a systematic use of scaling and applications of the implicit function theorem.

In this chapter, we use the same techniques as Kopell and Howard to study (4.1.1) when the nonlinearities are cubic. Our results confirm the conjecture made by Takens [49] on the bifurcation set of (4.1.1).

The results on quadratic nonlinearities are given in Section 9 in the form of exercises. Most of these results can be found in Kopell and Howard [39].

4.2. Preliminaries

Consider equation (4.1.1) where $A(\varepsilon)$ is given by (4.1.2). We also suppose that the linearization of $f(x,\varepsilon)$ is $A(\varepsilon)x$, and that

$$f(0,\varepsilon) \equiv 0, \; f(x,\varepsilon) \equiv -f(-x,\varepsilon). \qquad (4.2.1)$$

The object is to study the behavior of all small solutions of (4.1.1) for ε in a full neighborhood of the origin. Equation (4.1.1) is still too general, however, so we shall make some additional hypotheses on the nonlinear terms. Set $f = (f_1,f_2)^T$,

$$\alpha = \frac{\partial^3}{\partial x_1^3} f_2(0,0), \quad \beta = \frac{\partial^3}{\partial x_1^2 x_2} f_2(0,0).$$

(H1) $\alpha \neq 0$

(H2) $\beta \neq 0$

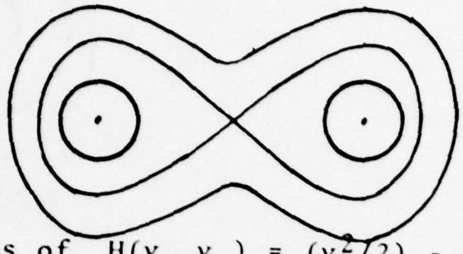
(H3) $\dfrac{\partial^3}{\partial x_1^3} f_1(0,0) = 0.$

(H1) implies that for small ε, (4.1.1) has either 1 or 3 fixed points. Under (H1), it is easy to show that by a change of co-ordinates in (4.1.1), we can assume (H3) (see Remark 1). We assume (H3) in order to simplify the computations.

Under (H1) we can prove the existence of families of periodic orbits and homoclinic orbits. Under (H1)-(H3) we can say how many periodic orbits of (4.1.1) exist for fixed ε. The sign of β will determine the direction of bifurcation and the stability of the periodic orbits among other things.

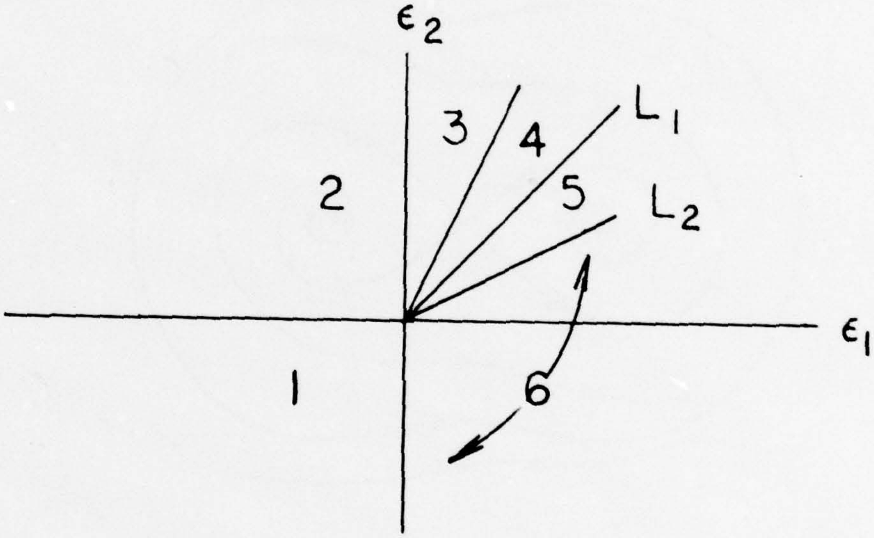
From now on we assume (H1)-(H3).

The main results are given in Figures 2-5. Sections 3-8 of this chapter show how we obtain these pictures. The pictures for $\beta > 0$ are obtained by using the change of variables $x_2 \to -x_2$, $\varepsilon_2 \to -\varepsilon_2$ and $t \to -t$.



Level Curves of $H(y_1,y_2) = (y_2^2/2) - (y_1^2/2) + (y_1^4/4)$.

FIG. 1


ϵ_2
3
4
L_1
2
5
L_2
ϵ_1
1
6

Bifurcation Set for the Case $\alpha < 0$, $\beta < 0$.

FIG. 2

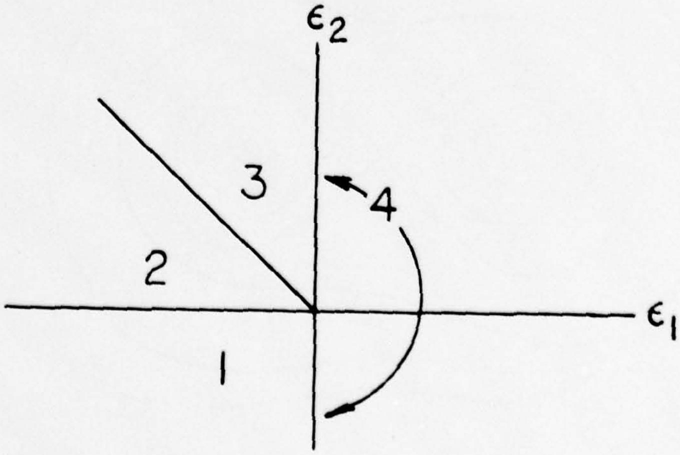
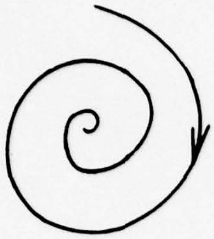
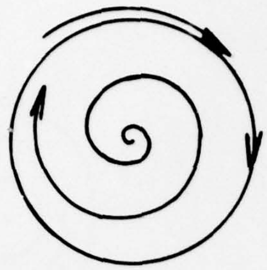

ϵ_2
3
4
2
ϵ_1
1

FIG. 3

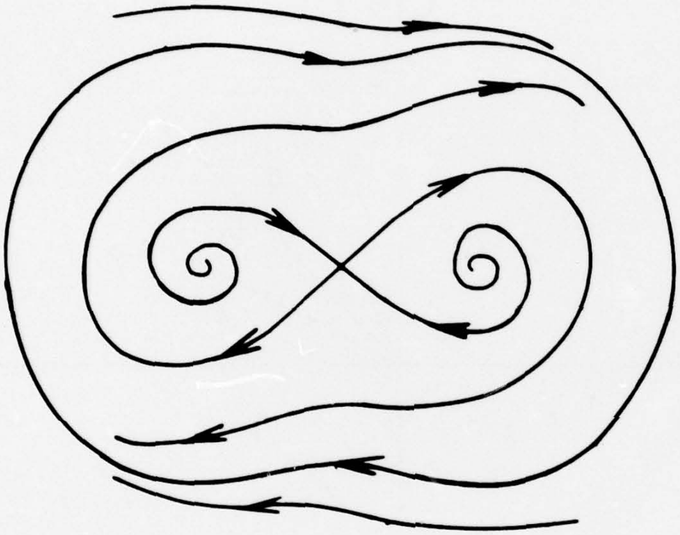
Bifurcation Set for the Case $\alpha > 0$, $\beta < 0$.



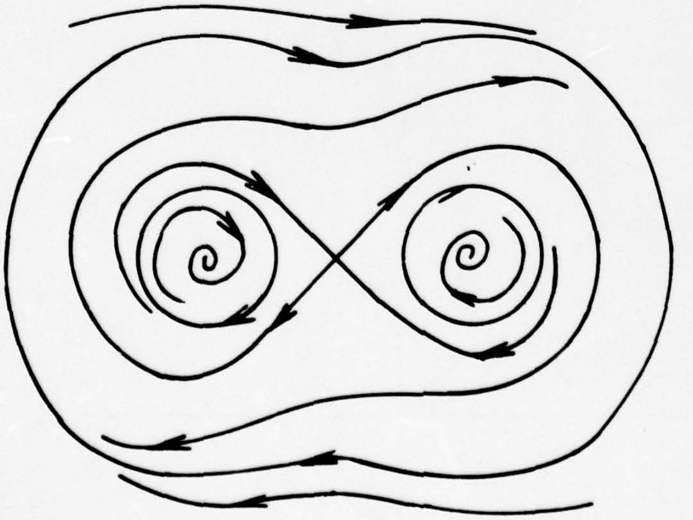
REGION 1



REGION 2



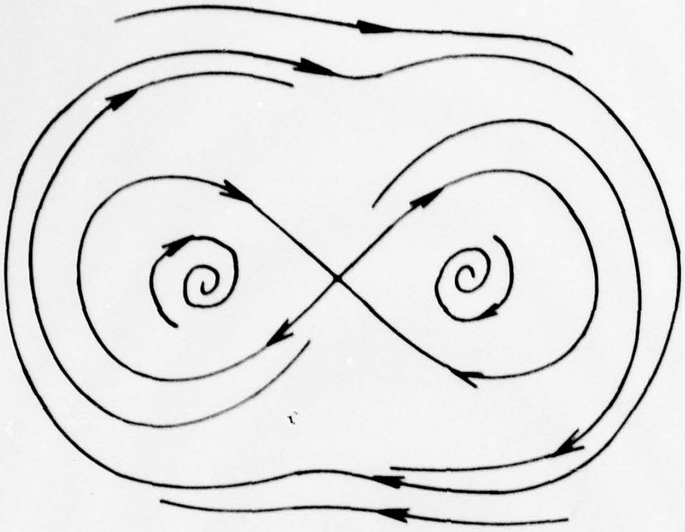
REGION 3



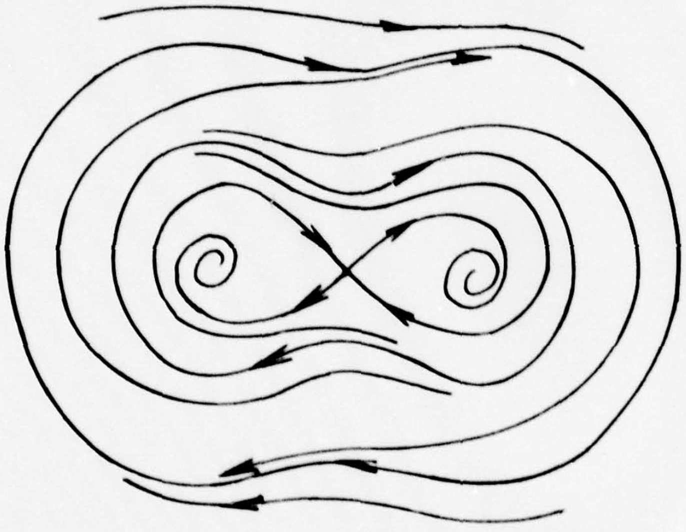
REGION 4

FIG. 4

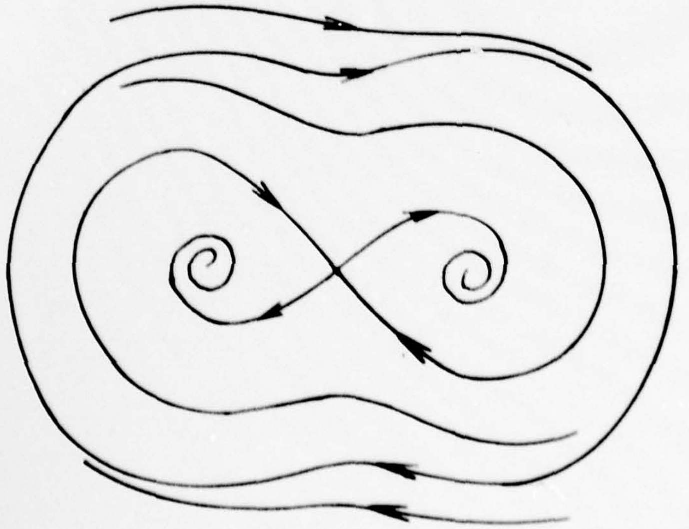
Phase Portraits for the Case $\alpha < 0$, $\beta < 0$.



ON L_1



REGION 5



ON L_2

FIG. 4 (CONT.)

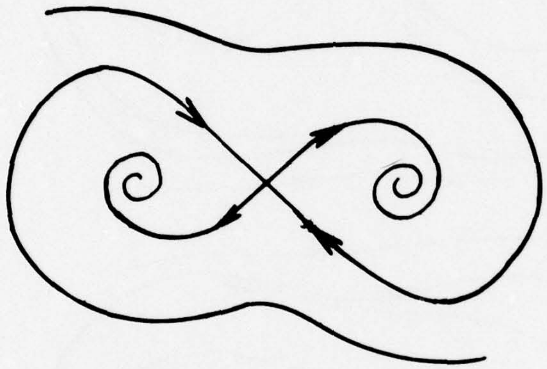


FIG 4. (CONT.)

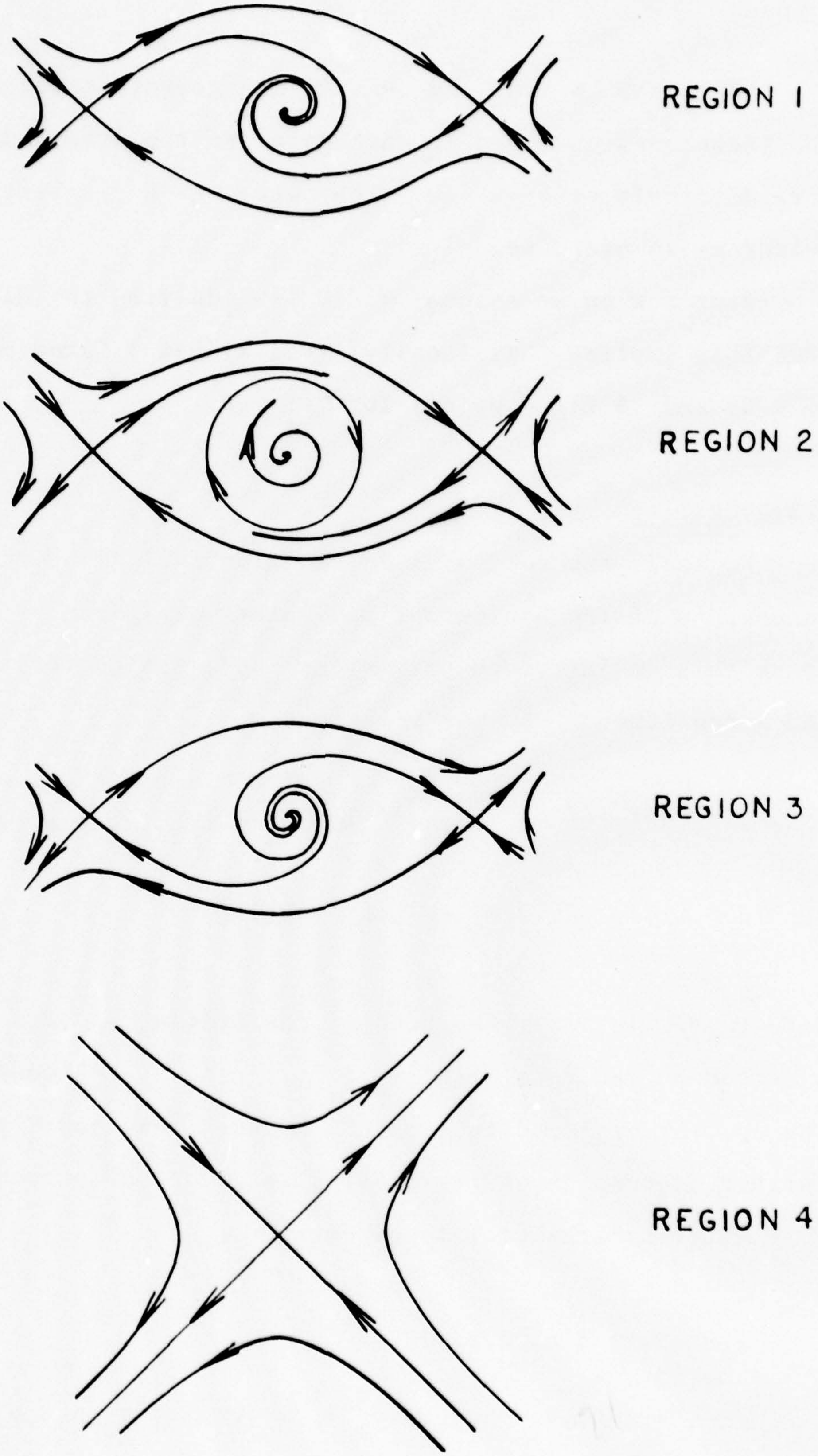

REGION 1
REGION 2
REGION 3
REGION 4

FIG. 5
Phase Portraits for the Case $\alpha > 0$, $\beta < 0$.

The cases $\alpha > 0$ and $\alpha < 0$ are geometrically different. The techniques involved in each case are the same and we only do the more difficult case $\alpha < 0$. The case $\alpha > 0$ is left for the reader as an exercise.

From now on we assume $\alpha < 0$ in addition to (H1)-(H3). Note that this implies that locally, (4.1.1) has 1 fixed point for $\varepsilon_1 < 0$ and 3 fixed points for $\varepsilon_1 > 0$.

4.3. Scaling

We scale the variables in equation (4.1.1) so that the first components of the non-zero fixed points are given by $\pm 1 + O(\varepsilon)$. To do this we introduce parameters μ, δ, scaled variables y_1, y_2 and a new time $\bar{t}$ by the relations

$$\delta = |\varepsilon_1 \alpha^{-1}|^{1/2},\ \varepsilon_2 = |\alpha|^{1/2}\delta\mu,\ x_1 = \delta y_1,\ x_2 = \delta^2|\alpha|^{1/2}y_2,$$

$$t = |\alpha|^{-1/2}\delta^{-1}\bar{t}.$$

For (μ,δ) in a neighborhood of the origin, $(\varepsilon_1,\varepsilon_2)$ belongs to a region of the form $\{(\varepsilon_1,\varepsilon_2): |\varepsilon_1| < \varepsilon_0,\ \varepsilon_1 \leq (\text{constant})\ \varepsilon_2^2\}$. The y_i are assumed to lie in a bounded set, say $|y_i| \leq M$. A further discussion of the scaling is given in Section 6.

After scaling (4.1.1) becomes

$$\begin{aligned} \dot{y}_1 &= y_2 + \delta^2 g_1(\mu,\delta,y) \\ \dot{y}_2 &= \operatorname{sgn}(\varepsilon_1)y_1 + \mu y_2 - y_1^3 + \delta\gamma y_1^2 y_2 + \delta^2 g_2(\mu,\delta,y) \end{aligned} \tag{4.3.1}$$

where the dot means differentiation with respect to $\bar{t}$, $\gamma = \beta|\alpha|^{-1/2}$ and $g_i(\mu,\delta,y) = O(1)$. The size of the bounds on the g_i depends on M,μ,δ. We write t for $\bar{t}$ from now on.

The cases $\varepsilon_1 > 0$ and $\varepsilon_1 < 0$ are treated separately.

4.4. The Case $\varepsilon_1 > 0$

With $\varepsilon_1 > 0$, equation (4.3.1) becomes

$$\begin{aligned} \dot{y}_1 &= y_2 + \delta^2 g_1(\mu,\delta,y) \\ \dot{y}_2 &= y_1 + \mu y_2 - y_1^3 + \delta\gamma y_1^2 y_2 + \delta^2 g_2(\mu,\delta,y). \end{aligned} \tag{4.4.1}$$

Let $H(y_1,y_2) = (y_2^2/2) - (y_1^2/2) + (y_1^4/4)$. Then along solutions of (4.4.1),

$$\dot{H}(y_1,y_2) = \mu y_2^2 + \delta\gamma y_1^2 y_2^2 + O(\delta^2). \tag{4.4.2}$$

Note that for $\mu = \delta = 0$, H is a first integral of (4.4.1).

The level curves of $H(y_1,y_2) = b$ consist of a figure of eight if $b = 0$, and a single closed curve if $b > 0$ (see Figure 1). For $b \geq 0$, the curve $H(y_1,y_2) = b$ passes through the point $y_1 = 0$, $y_2 = (2b)^{1/2}$. For $b \geq 0$ and δ sufficiently small, we prove the existence of a function $\mu = \mu_1(b,\delta) = -\gamma P(b)\delta + O(\delta^2)$. For $b > 0$, (4.4.1) with $\mu = \mu_1(b,\delta)$ has a periodic solution passing through the point $y_1 = 0$, $y_2 = (2b)^{1/2}$. With $\mu = \mu_1(0,\delta)$, (4.4.1) has a figure of eight solution.

For fixed μ,δ, the number of periodic solutions of (4.4.1), surrounding all three fixed points, depends upon the number of solutions of

$$\mu = \mu_1(b,\delta) = -\gamma P(b)\delta + O(\delta^2). \qquad (4.4.3)$$

We prove that $P(b) \to \infty$ as $b \to \infty$ and that there exists $b_1 > 0$ such that $P'(b) < 0$ for $b < b_1$ and $P'(b) > 0$ for $b > b_1$. These properties of $P(b)$ determine the number of solutions of (4.4.3).

Suppose, for simplicity of exposition, that $\mu_1(b,\delta) \equiv -\gamma P(b)\delta$ and that $\gamma < 0$. If $0 < b_2 < b_1$, then there exists $b_3 > b_1$ such that $\mu_1(b_2,\delta) \equiv \mu_1(b_3,\delta)$. Hence, if $\mu = \mu_1(b_2,\delta)$, then (4.4.1) has two periodic solutions, one passing through $y_1 = 0$, $y_2 = (2b_2)^{1/2}$, the other passing through $y_1 = 0$, $y_2 = (2b_3)^{1/2}$. If $\mu > \mu_1(0,\delta)$, then (4.4.1) has one periodic solution surrounding all three fixed points. Finally, if $\mu = \mu(b_1,\delta)$, then the periodic solutions coincide.

In Figure 4, the periodic solutions surrounding all three fixed points in regions 3-5 correspond to the periodic solutions of (4.4.1) which are parametrized by $\mu = \mu_1(b,\delta)$, $b > b_1$. Similarly the "inner" periodic solutions in region 5 are parametrized by $\mu = \mu_1(b,\delta)$, $0 < b < b_1$. The curve L_1 in $(\varepsilon_1,\varepsilon_2)$ space corresponds to the curve $\mu = \mu_1(0,\delta)$. Similarly the curve L_2 corresponds to the curve $\mu = \mu_1(b_1,\delta)$ (see Figure 2).

In general $\mu_1(b,\delta)$ is not identically equal to $-\gamma P(b)\delta$, but the results are qualitatively the same. For example, we prove the existence of a function $b_1(\delta) = b_1 + O(\delta)$, such that if μ,δ satisfy $\mu = \mu_1(b_1(\delta),\delta)$, then equation (4.4.3) has exactly one solution. The curve $\mu = \mu_1(b_1(\delta),\delta)$ which is mapped into the curve L_2 in $(\varepsilon_1,\varepsilon_2)$ space, corresponds to the points where the two periodic solutions coincide.

If $b < 0$, then the set of points for which $H(y_1,y_2) = b$ consists of two closed curves surrounding the points $(-1,0)$ and $(1,0)$. For $0 < c < 1$, we prove the existence of a function $\mu = \mu_2(c,\delta) = -\gamma Q(c)\delta + O(\delta^2)$. For $\mu = \mu_2(c,\delta)$, (4.4.1) has a periodic solution surrounding the point $(1,0)$ and passing through $y_1 = c$, $y_2 = 0$. Using $f(x) = -f(-x)$, this proves the existence of a periodic solution surrounding the point $(-1,0)$ and passing through $y_1 = -c$, $y_2 = 0$.

We also prove that $Q'(c) > 0$ for $0 < c < 1$. Let $\delta > 0$ and suppose μ satisfies

$$\mu_2(0,\delta) < -\mathrm{sgn}(\gamma)\mu < \mu(1,\delta). \qquad (4.4.4)$$

Then the equation $\mu = \mu_2(c,\delta)$ has exactly one solution. Hence, for fixed μ,δ satisfying (4.4.4), equation (4.4.1) has exactly one periodic solution surrounding $(1,0)$. The region in (μ,δ) space, corresponding to (4.4.4), is mapped into region 4 in $(\varepsilon_1,\varepsilon_2)$ space (see Figure 2).

Lemma 1. For δ sufficiently small, there exists a function $\mu = \mu(\delta) = (4/5)\delta + O(\delta^2)$ such that when $\mu = \mu(\delta)$, (4.4.1) has a homoclinic orbit.

Proof. Let $S(\mu,\delta)$, $U(\mu,\delta)$ be the stable and unstable manifolds of the fixed point $(0,0)$ in (4.4.1). These manifolds exist, since for μ,δ sufficiently small, $(0,0)$ is a saddle [23]. Let $H(\mu,\delta,+)$ be the value of $H(y_1,y_2)$ when $U(\mu,\delta)$ hits $y_2 = 0$, $y_1 > 0$. Similarly, $H(\mu,\delta,-)$ is the value of $H(y_1,y_2)$ when $S(\mu,\delta)$ hits $y_2 = 0$, $y_1 > 0$. $H(\mu,\delta,\pm)$ are well defined since stable and unstable manifolds depend continuously on parameters.

Let $I(\mu,\delta,+)$ denote the integral of $\dot{H}(y_1,y_2)$ over the portion of $U(\mu,\delta)$ with $y_1 > 0$, $y_2 > 0$ from $y_1 = y_2 = 0$ to $y_2 = 0$, $y_1 > 0$. Then

$$H(\mu,\delta,+) = I(\mu,\delta,+). \tag{4.4.5}$$

Similarly, $I(\mu,\delta,-)$ denotes the integral of $\dot{H}(y_1,y_2)$ over the portion of $S(\mu,\delta)$ with $y_1 > 0$, $y_2 < 0$ from $y_1 = y_2 = 0$ to $y_1 = 0$, $y_2 > 0$, so that $H(\mu,\delta,-) = I(\mu,\delta,-)$.

Equation (4.4.1) has a homoclinic orbit (with $y_1 > 0$) if and only if

$$H(\mu,\delta,+) - H(\mu,\delta,-) = 0. \tag{4.4.6}$$

We solve (4.4.6) by the implicit function theorem

Using (4.4.2) and (4.4.5),

$$H(\mu,\delta,+) = \int_{+} (\mu y_2^2 + \gamma\delta y_1^2 y_2^2)\,dt + O(\mu^2 + \delta^2), \tag{4.4.7}$$

where the above integral is taken over the portion of $U(0,0)$ from $y_1 = y_2 = 0$ to $y_1 = 0$, $y_2 > 0$. Similarly,

$$H(\mu,\delta,-) = \int_{-} (\mu y_2^2 + \gamma\delta y_1^2 y_2^2)dt + O(\mu^2 + \delta^2), \qquad (4.4.8)$$

where the integral is taken over the portion of $S(0,0)$ from $y_1 = y_2 = 0$ to $y_1 = 0,\ y_2 > 0$. Using (4.4.7), (4.4.8) and $U(0,0) = -S(0,0)$, we obtain

$$H(\mu,\delta,+) = -H(\mu,\delta,-) + O(\mu^2 + \delta^2) \qquad (4.4.9)$$

Using (4.4.7) and (4.4.9),

$$\frac{\partial}{\partial\mu} (H_1(0,0,+) - H_1(0,0,-)) = 2 \int_{+} y_2^2 dt > 0,$$

so that by the implicit function theorem, we can solve (4.4.6) for μ as a function of δ, say $\mu = \mu(\delta)$. We now show how to get an approximate formula for $\mu(\delta)$. We can write equation (4.4.6) in the form

$$\mu \int_{+} y_2^2 dt + \gamma\delta \int_{+} y_1^2 y_2^2 dt + O(\mu^2 + \delta^2) = 0.$$

Hence, using (4.4.1) and $y_2 = +[y_1^2 - (y_1^4/2)]^{1/2}$, we obtain

$$\mu = \frac{-\gamma\delta \int_{+} y_1^2 y_2 dy_2}{\int_{+} y_2 dy_1} + O(\delta^2) = -\frac{4\gamma\delta}{5} + O(\delta^2)$$

This completes the proof of Lemma 1.

Lemma 1 proves the existence of a homoclinic orbit of (4.1.1) when $\varepsilon_1, \varepsilon_2$ lie on the curve L_1 given by

$$\varepsilon_2 = -(4/5)|\alpha|^{-1}\beta\varepsilon_1 + O(\varepsilon_1^{3/2}).$$

Using $f(x,\varepsilon) = -f(-x,\varepsilon)$, when $\varepsilon_1, \varepsilon_2$ lie on L_1, equation (4.1.1) has a figure of eight solution.

We now prove the existence of periodic solutions of (4.4.1) surrounding all three fixed points. In the introduction to this section, we stated that (4.4.1) has a periodic solution passing through $y_1 = 0$, $y_2 = (2b)^{1/2}$ for any $b > 0$. In Lemma 2, we only prove this for "moderate" values of b. The reason for this is that in (4.3.1) the g_i are bounded only for y in a bounded set. In Section 6, we show that by a simple modification of the scaling, we can extend these results to all $b > 0$.

Lemma 2. Fix $\bar{b} > 0$. Then for $0 < b < \bar{b}$ and δ sufficiently small, there exists a function $\mu = \mu_1(b,\delta) = -\gamma P(b)\delta + O(\delta^2)$. If $\mu = \mu_1(b,\delta)$ in (4.4.1), then (4.4.1) has a periodic solution passing through $y_1 = 0$, $y_2 = (2b)^{1/2}$. As $b \to 0$ the periodic solution tends to the figure of eight solution obtained in Lemma 1.

Proof. Let $H(\mu,\delta,b,+)$ be the value of $H(y_1,y_2)$ when the orbit of (4.4.1), which starts at $y_1 = 0$, $y_2 = (2b)^{1/2}$, intersects $y_2 = 0$. Similarly, $H(\mu,\delta,b,-)$ is the value of $H(y_1,y_2)$ when the orbit of (4.4.1), which starts at $y_1 = 0$, $y_2 = -(2b)^{1/2}$, is integrated backwards in time until it intersects $y_2 = 0$. Then (4.4.1) has a periodic solution passing through $y_1 = 0$, $y_2 = (2b)^{1/2}$ if and only if

$$H(\mu,\delta,b,+) - H(\mu,\delta,b,-) = 0. \qquad (4.4.10)$$

Let $I(\mu,\delta,b,+)$ denote the integral of $\dot{H}(y_1,y_2)$ over the portion of the orbit of (4.4.1) with $y_2 > 0$, starting at $y_1 = 0$, $y_2 = (2b)^{1/2}$ and finishing at $y_2 = 0$, $y_1 > 0$. Similarly, $I(\mu,\delta,b,-)$ is defined by integrating backwards in time. Thus,

$$H(\mu,\delta,b,\pm) = b + I(\mu,\delta,b,\pm). \qquad (4.4.11)$$

Using (4.4.2) and (4.4.11),

$$H(\mu,\delta,b,+) = b + \int (\mu y_2^2 + \gamma\delta y_2^2 y_1^2)dt + O(\mu^2 + \delta^2) \qquad (4.4.12)$$

where the above integral is taken over the portion of the orbit of (4.4.1) with $\mu = \delta = 0$ from $y_1 = 0$, $y_2 = (2b)^{1/2}$ to $y_1 = c$, $y_2 = 0$ where

$$4b = c^4 - 2c^2. \qquad (4.4.13)$$

Similarly, $H(\mu,\delta,b,-) = -H(\mu,\delta,b,+) + O(\mu^2 + \delta^2)$, so that equation (4.4.10) may be written in the form

$$\int (\mu y_2^2 + \gamma\delta y_1^2 y_2^2)dt + O(\mu^2 + \delta^2) = 0. \qquad (4.4.14)$$

Hence by the implicit function theorem, we can solve (4.4.14) to obtain $\mu = -\gamma P(b)\delta + O(\delta^2)$ where

$$P(b) = \frac{\int y_1^2 y_2 dy_1}{\int y_2 dy_1}. \qquad (4.4.15)$$

In order to prove that the periodic solution tends to the figure of eight solution as $b \to 0$, we prove that

$$H(\mu,\delta,b,\pm) \to H(\mu,\delta,\pm) \quad \text{as} \quad b \to 0. \qquad (4.4.16)$$

This does not follow from continuous dependence of solutions on initial conditions, since as $b \to 0$ the period of the periodic solution tends to infinity. The same problem occurs in Kopell and Howard [39, p. 339] and we outline their method. For y_1 and y_2 small, solutions of (4.4.1) behave like solutions of the linearized equations. The proof of (4.4.16) follows from the fact that the periodic solution stays close to the solution of the linearized equation for the part of the solution with y_1, y_2 small and continuous dependence on initial data for the rest of the solution.

Lemma 3. $P(b) \to \infty$ as $b \to \infty$.

Proof. The integrals in (4.4.15) are taken over the curve $y_2 = [y_1^2 - (y_1^4/2) + 2b]^{1/2}$, from $y_1 = 0$ to $y_1 = c$ where c is defined by (4.4.13). Thus, $J_0(b)P(b) = J_1(b)$ where

$$J_i(b) = \int_0^c w^{2i}(w^2 - (w^4/2) + 2b)^{1/2}dw. \qquad (4.4.17)$$

Substituting $w = cz$ in (4.4.17) we obtain

$$J_i(b) = c^2 \int_0^1 (cz)^{2i}g(z)dz$$

where $g(z) = [(z^2-1) + (c^2/2)(1-z^4)]^{1/2}$. Since $g(z) \le g(c^{-1})$ for $0 \le z \le 1$, we have that $J_0(b) \le D_1c^3$ for some positive constant D_1. Similarly, there exists a positive constant D_2 such that $J_1(b) \ge D_2c^5$. The result now follows.

Lemma 4. There exists $b_1 > 0$ such that $P'(b) < 0$ for $b < b_1$ and $P'(b) > 0$ for $b > b_1$.

Proof. It is easy to show that $P'(b) = -\infty$ as $b \to 0$. Hence, by Lemma 3 it is sufficient to show that if $P'(b_1) = 0$ then $P''(b_1) > 0$.

Let $r(w) = [w^2 - (w^4/2) + 2b]^{1/2}$. Differentiating (4.4.17) with respect to b we obtain

$$J_i' = \int_0^c \frac{w^{2i}}{r(w)}\, dw. \qquad (4.4.18)$$

Integrating by parts in J_0 we obtain

$$J_0 = \int_0^c \frac{(w^4-w^2)}{r(w)}\, dw. \tag{4.4.19}$$

Also

$$J_0 = \int_0^c \frac{r^2(w)}{r(w)}\, dw = \int_0^c \frac{[w^2 - (w^4/2) + 2b]}{r(w)}\, dw. \tag{4.4.20}$$

Similarly,

$$3J_1 = \int_0^c \frac{(w^6-w^4)}{r(w)}\, dw = 3\int_0^c \frac{w^2}{r(w)}\, [w^2 - (w^4/2) + 2b]dw. \tag{4.4.21}$$

Using (4.4.18) - (4.4.21) we can express J_0 and J_1 in terms of J_0' and J_1'. A straightforward calculation yields

$$\begin{aligned} 3J_0 &= 4bJ_0' + J_1' \\ 15J_1 &= 4bJ_0' + (4+12b)J_1'. \end{aligned} \tag{4.4.22}$$

Suppose that $P'(b_1) = 0$. Then $J_0(b_1)P''(b_1) = J_1''(b_1) - P(b_1)J_0''(b_1)$. Using (4.4.22) we obtain

$$4b_1(4b_1+1)[J_1''(b_1) - P(b_1)J_0''(b_1)] = J_0'(b_1)[P^2(b_1) + 8b_1P(b_1) - 4b_1]. \tag{4.4.23}$$

Hence $P''(b_1)$ has the same sign as

$$P^2(b_1) + 8b_1P(b_1) - 4b_1. \tag{4.4.24}$$

Since $P'(b_1) = 0$ we have that $J_1'(b_1) = P(b_1)J_0'(b_1)$. Using (4.4.22) we obtain

$$5P^2(b_1) + 8b_1P(b_1) - 4P(b_1) - 4b_1 = 0. \tag{4.4.25}$$

Using $b_1 > 0$, it is easy to show that (4.4.25) implies that $P(b_1) < 1$. Using (4.4.24) and (4.4.25), $P''(b_1)$ has the same sign as $P(b_1) - P^2(b_1)$. This proves that $P''(b_1) > 0$ as required.

Lemma 5. For δ sufficiently small there exist $b_1(\delta) = b_1 + O(\delta)$, $b_2(\delta) = b_2 + O(\delta)$, where $P(0) = P(b_2)$, with the following properties: Let $\delta > 0$.

(i) If $\mu = \mu_1(b_1(\delta),\delta)$, then the equation

$$\mu = \mu_1(b,\delta) \tag{4.4.26}$$

has exactly one solution.

(ii) If $\gamma < 0$ and $\mu_1(b_1(\delta),\delta) < \mu < \mu_1(b_2(\delta),\delta)$, then equation (4.4.26) has exactly two solutions $b_3(\delta)$, $b_4(\delta)$ with $b_3(\delta) = b_3 + O(\delta)$, $b_4(\delta) = b_4 + O(\delta)$, where b_3 and b_4 are solutions of

$$P(b) = -\mu\delta^{-1}\gamma^{-1}. \tag{4.4.27}$$

A similar result holds if $\gamma > 0$.

(iii) If $\gamma < 0$ and $\mu > \mu_1(b_2(\delta),\delta)$, then (4.4.26) has exactly one solution $b_5(\delta) = b_5 + O(\delta)$, where b_5 is the unique solution of (4.4.27).

(iv) If $\gamma < 0$ and $\mu < \mu_1(b_1(\delta),\delta)$, then (4.4.27) has no solutions. A similar result holds for $\gamma > 0$.

Proof. By Lemma 4, there exists $b_2 > 0$ such that $P(b_2) = P(0)$ and $P'(b_2) > 0$. Set $g(z,\delta) = \delta^{-1}[\mu_1(b_2+z,\delta) - \mu_1(0,\delta)]$, for $\delta \neq 0$ and $g(z,0) = 0$. Then $g(z,\delta) = P'(b_2)z + O(|\delta| + z^2)$. By the implicit function theorem there exists $z(\delta) = O(\delta)$ such that $g(z(\delta),\delta) = 0$. Hence, if $b_2(\delta) = b_2 + z(\delta)$ then $\mu_1(b_2(\delta),\delta) = \mu_1(0,\delta)$. The existence of $b_1(\delta)$ is proved in a similar way. The rest of the Lemma follows from the properties of $P(b)$.

We now prove the existence of periodic solutions of (4.4.1) surrounding a single fixed point.

Lemma 6. For $0 < c < 1$ and δ sufficiently small, there exists $\mu = \mu_2(c,\delta) = -\gamma Q(c)\delta + O(\delta^2)$. If $\mu = \mu_2(c,\delta)$, then (4.4.1) has a periodic solution passing through $y_1 = c$, $y_2 = 0$. As $c \to 0$ the periodic orbits tend to the homoclinic orbit obtained in Lemma 1.

Proof. Let $H(\mu,\delta,c,+)$ be the value of $H(y_1,y_2)$ when the orbit of (4.4.1) starting at $y_1 = c$, $y_2 = 0$ intersects $y_2 = 0$, $\dot{y}_2 > 0$. Similarly, $H(\mu,\delta,c,-)$ is the value of $H(y_1,y_2)$ when the orbit of (4.4.1) starting at $y_1 = c$, $y_2 = 0$ is integrated backwards in time until it intersects $y_2 = 0$. Equation (4.4.1) will have a periodic orbit passing through $y_1 = c$, $y_2 = 0$ if and only if

$$H(\mu,\delta,c,+) - H(\mu,\delta,c,-) = 0. \tag{4.4.28}$$

Using the same method as in Lemma 2, we can rewrite equation (4.4.28) as

$$G(\mu,\delta,c) = \int (\mu y_2^2 + \gamma\delta y_1^2 y_2^2)dt + O(\mu^2 + \delta^2) = 0 \tag{4.4.29}$$

where the above integral is taken over the curve

$$y_2 = r(y_1) = [y_1^2 - (y_1^4/2) + (c^4/2) - c^2]^{1/2}, \tag{4.4.30}$$

from $y_1 = c$, $y_2 = 0$ to $y_2 = 0$ again. By (4.4.29)

$$\frac{\partial}{\partial\mu} G(0,0,c) = \int y_2^2 dt > 0. \tag{4.4.31}$$

Thus, for fixed c we can solve (4.4.29). We cannot solve (4.4.29) uniformly in c however, since as $c \to 1$ the right hand side of (4.4.31) tends to 0. We use a method similar to that used by Kopell and Howard in [39, p. 337-338] to obtain $\mu_2(c,\delta)$ for $0 < c < 1$.

Equation (4.4.1) has a fixed point at $(y_1,y_2) = (1,0) + O(|\mu|,|\delta|)$. For μ,δ sufficiently small we make a change of variables $\bar{y}_j = h_j(y_1,y_2,\mu,\delta)$ so that the fixed point is transformed into $(\bar{y}_1,\bar{y}_2) = (1,0)$. An easy calculation shows that if we make this transformation, then the only change in (4.4.1) is in the functions g_1 and g_2. We suppose that the above change of variables has been made and we write y_j for $\bar{y}_j$.

The curves $H(y_1,y_2) = H(c,0)$ can be written in the form

$$4H(y_1,y_2) + 1 = 2y_2^2 + (y_1-1)^2(y_1+1)^2 = (c-1)^2(c+1)^2. \tag{4.4.32}$$

Thus, for c close to 1 the closed curves are approximately

$$y_2^2 + 2(y_1-1)^2 = 2(c-1)^2. \tag{4.4.33}$$

so instead of equation (4.4.30) we consider the equation

$$\bar{G}(\mu,\delta,c) = (c-1)^{-2}G(c,\mu,\delta) = 0. \tag{4.4.34}$$

If we prove that $\bar{G}(\mu,\delta,c)$ is bounded for $0 < c < 1$ and that $(\partial/\partial\mu)\bar{G}(0,0,c)$ is bounded away from zero for $0 < c < 1$, then we can solve (4.4.34) uniformly in c.

Now

$$\dot{H}(y_1,y_2) = \mu y_2^2 + \delta[\gamma y_1^2y_2^2 + \delta y_2g_2 + \delta(y_1^3-y_1)g_1].$$

Since the fixed point is at $(1,0)$, $\dot{H}(y_1,y_2)$ is bounded above and below by quadratic forms in y_2 and $y_1 - 1$. By (4.4.32), $4H(y_1,y_2) + 1$ is bounded above and below by quadratic forms in y_2

and $y_1 - 1$. Hence there exist functions $q_i(\mu,\delta) = O(|\mu| + |\delta|)$ such that

$$q_1(\mu,\delta) \leq \frac{d}{dt} \ln(4H(y_1,y_2) + 1) \leq q_2(\mu,\delta).$$

Integrating the above inequality over the curve given by (4.4.30) we obtain

$$\exp(q_1(\mu,\delta)T) \leq [4H(\mu,\delta,c,+) + 1]/[4H(c,0) + 1] \leq \exp(q_2(\mu,\delta)T) \qquad (4.4.35)$$

where T is a bound for the time taken to trace the orbit. Using (4.4.29), (4.4.32) and (4.3.35) we see that $\overline{G}(\mu,\delta,c)$ is bounded. The fact that $(\partial/\partial\mu)\overline{G}(c,0,0)$ is bounded away from zero follows easily from (4.4.32). This proves that for δ sufficiently small and $0 < c < 1$, we can solve (4.4.32) to obtain $\mu = \mu_2(c,\delta) = -\gamma Q(c)\delta + O(\delta^2)$ where $J_0(c)Q(c) = J_1(c)$,

$$J_i(c) = \int_c^d w^{2i} r(w)\,dw$$

and $r(w)$ is defined by (4.4.30) and $r(d) = 0$, $d > c$.

The fact that the periodic solution tends to the homoclinic orbit is proved in the same way as the corresponding result in Lemma 2.

Using $f(x) = -f(-x)$, Lemma 6 proves the existence of periodic solutions of (4.4.1) surrounding $(-1,0)$.

Lemma 7. $Q'(c) > 0$ for $0 < c < 1$.

Proof. We write Q, J_0 and J_1 as functions of b where $4b = c^4 - 2c^2$. Since $(db/dc) < 0$ for $0 < c < 1$, we must prove that $Q'(b) < 0$ for $-1 < 4b < 0$. Following the same procedure as in Lemma 4, we find that J_0, J_0', J_1, J_1' satisfy equation (4.4.22). Thus, if $Q'(b_1) = 0$ then

$$5Q^2(b_1) + 8b_1Q(b_1) - 4Q(b_1) - 4b_1 = 0. \qquad (4.4.36)$$

Since $-1 < 4b_1 < 0$, the roots of (4.4.36) are less than 1. Hence, if $Q'(b_1) = 0$ then $Q(b_1) < 1$. Also, from (4.4.23), if $Q'(b_1) = 0$ then $Q''(b_1)$ has the same sign as $4b_1 - 8b_1Q(b_1) - Q^2(b_1)$. Using (4.4.36) and $Q(b_1) < 1$, this implies that $Q''(b_1) < 0$. Since $Q(-1/4) = 1$, $Q(0) = 4/5$, this shows that $Q'(b) < 0$ for $-1 < 4b < 0$. This completes the proof of the Lemma.

4.5. The Case $\varepsilon_1 < 0$

With $\varepsilon_1 < 0$, equation (4.3.1) becomes

$$\begin{aligned} \dot{y}_1 &= y_2 + \delta^2 g_1(\mu,\delta,y) \\ \dot{y}_2 &= -y_1 + \mu y_2 - y_1^3 + \delta\gamma y_1^2 y_2 + \delta^2 g_2(\mu,\delta,y). \end{aligned} \qquad (4.5.1)$$

Let $H_1(y_1,y_2) = (y_2^2/2) + (y_1^2/2) + (y_1^4/4)$. Then along solutions of (4.5.1), $\dot{H}_1 = \mu y_2^2 + \delta\gamma y_1^2 y_2^2 + O(\delta^2)$. Using the same methods as in the previous Lemmas, we prove that (4.5.1) has a periodic solution passing through $y_1 = c > 0$, $y_2 = 0$, if and only if $\mu = \mu_3(c,\delta) = -\gamma R(c)\delta + O(\delta^2)$, where $J_0(c)R(c) = J_1(c)$,

$$J_i(c) = \int_0^c w^{2i} r(w)\,dw,$$

$$r(w) = [2b - (w^4/2) - w^2]^{1/2},\ 4b = c^4 + 2c^2.$$

In order to prove that for fixed μ,δ, equation (4.5.1) has at most one periodic solution we prove that R is strictly monotonic.

Lemma 8. $R'(c) > 0$ for $c > 0$.

Proof. We write R, J_0, J_1 as functions of b. It is sufficient to prove that $R'(b) > 0$ for $b > 0$.

Using the same methods as before, we show that

$$\begin{aligned} 3J_0 &= 4bJ_0' - J_1' \\ 15J_1 &= (12b+4)J_1' - 4bJ_0' \end{aligned} \tag{4.5.2}$$

$$\int_0^c \frac{3w^4\,dw}{r(w)} = 4bJ_0' - 4J_1' . \tag{4.5.3}$$

Now R' has the same sign as S where $S = 15[J_1'J_0-J_0'J_1]$. By (4.5.2),

$$S = (8b-4)J_0'J_1' = 5(J_1')^2 + 4b(J_0')^2.$$

By (4.5.3), $bJ_0' > J_1'$. Hence, for $b \geq 2$.

$$S \geq (J_1')^2b^{-1}(8b-4-5b+4) = 3(J_1')^2 > 0.$$

Similarly, if $0 < b < 2$, then $S > 3b^2(J_0')^2 > 0$. This completes the proof of Lemma 8.

4.6. More Scaling

In Section 4 we proved the existence of periodic solutions of (4.4.1) which pass through the point $(0,(2b)^{1/2})$. Lemma 3 indicates that we may take b to be as large as we please. However our analysis relied on the fact that the g_i are $O(1)$ and this is true only for y in a bounded set. Also, our analysis in Sections 4 and 5 restricts $\varepsilon_1,\varepsilon_2$ to a region of the form $\{(\varepsilon_1,\varepsilon_2): |\varepsilon_1| \leq \varepsilon_0, \varepsilon_1 \leq (\text{constant})\ \varepsilon_2^2\}$.

To remedy this we modify the scaling by setting $\delta = |\varepsilon,\alpha^{-1}|^{1/2}h^{-1}$, where h is a new parameter with $0 \leq 4h \leq 1$ say. The other changes of variables remain the same. Note that if μ,δ lie in a full neighborhood of the origin then $(\varepsilon_1,\varepsilon_2)$ and (x_1,x_2) lie in a full neighborhood of the origin.

After scaling, (4.1.1) becomes

$$\begin{aligned}\dot{y}_1 &= y_2 + \delta^2 g_1(\mu,\delta,h,y)\\ \dot{y}_2 &= h^2\operatorname{sgn}(\varepsilon_1)y_1 + \mu y_2 - y_1^3 + \delta\gamma y_1^2 y_2 + \delta^2 g_2(\mu,\delta,h,y).\end{aligned} \tag{4.6.1}$$

For $0 \le 4h \le 1$ and μ,δ sufficiently small, the g_i are $O(1)$ for y in a bounded set.

Let $\varepsilon_1 > 0$. Let $H_2(y_1,y_2) = (y_2^2/2) - (h^2y_1^2/2) + (y_1^4/4)$. Then along solutions of (4.6.1), $\dot{H}_2 = \mu y_2^2 + \gamma\delta y_1^2 y_2^2 + O(\mu^2+\delta^2)$. Following the same procedure as in Section 4, we find that (4.6.1) has a periodic solution passing through $y_1 = 0$, $y_2 = 1$, if and only if

$$\mu = h^2\mu_1(b,\delta) = -\gamma h^2 P(b)\delta + O(\delta^2),$$

where $2b = h^{-4}$. An easy computation shows that as $h \to 0$, $h^2P(b) = K + O(h^2)$, where $J_0K = \sqrt{2}\,J_1$,

$$J_i = \int_0^1 w^{2i}(1-w^4)^{1/2}dw.$$

Thus, for small h, (4.6.1) has a periodic solution passing through $y_1 = 0$, $y_2 = 1$ if and only if $\mu = -\gamma K\delta + O(h^2|\delta| + \delta^2)$. In particular, when $\varepsilon_1 = 0$, $\varepsilon_2 > 0$, $\beta < 0$, (4.1.1) has a periodic solution passing through $x_1 = 0$, $x_2 = |\alpha|(-\beta K)^{-1}\varepsilon_2 + O(\varepsilon_2^{3/2})$.

For $\varepsilon_1 < 0$, a similar anlysis shows that (4.6.1) has a periodic solution passing through $y_1 = 1$, $y_2 = 0$, if and only if

$$\mu = h^2\mu_2(h^{-1},\delta) = -\gamma h^2 Q(h^{-1})\delta + O(\delta^2),$$

and that as $h \to 0$, $h^2Q(h^{-1}) = (K/\sqrt{2}) + O(h^2)$.

4.7. Completion of the Phase Portraits

Our analysis in the previous sections proves the existence of periodic and homoclinic orbits of (4.1.1) for certain regions in $\varepsilon_1 - \varepsilon_2$ space. We now show how to complete the pictures.

Obtaining the complete phase portrait in the different regions involves many calculations. However, the method is the same in each case so we only give one representative example. We prove that if $\gamma < 0$, then the "outer" periodic orbits in region 5 is stable. We use the scaling given in Section 3.

Since we are in region 5, $\mu_1(0,\delta) > \mu > \mu_1(b_1(\delta),\delta)$. Fix μ and δ and let b_2 be the solution of $\mu = \mu_1(b,\delta)$ with $b_2 > b_1(\delta)$. Then we have to prove that the periodic solution Γ of (4.4.1) passing through $y_1 = 0$, $y_2 = (2b_2)^{1/2}$ is stable.

Let b_3 be the solution of $\mu = \mu_1(b,\delta)$ with $b_3 < b_1(\delta)$. Then there is a periodic solution passing through $y_1 = 0$, $y_2 = (2b_3)^{1/2}$. In fact, we prove that any solution of (4.4.1) starting "outside" this periodic solution (and inside some bounded set) tends to Γ as $t \to \infty$.

Let $b > b_3$. Let $c(b,+)$ be the value of y_1 when the orbit of (4.4.1) starting at $y_1 = 0$, $y_2 = (2b)^{1/2}$ first hits $y_2 = 0$. Similarly, $c(b,-)$ is the value of y_1 when the orbit of (4.1.1) starting at $y_1 = 0$, $y_2 = (2b)^{1/2}$ is integrated backwards until it hits $y_2 = 0$. Using $f(x,\varepsilon) = -f(-x,\varepsilon)$, the solution passing through $(0,(2b)^{1/2})$, spirals inwards or outwards according to the sign of

$c(b,+) + c(b,-)$. Using the calculation in Lemma 2, $c(b,+) + c(b,-)$ has the same sign as $H(\mu,\delta,b,+) - H(\mu,\delta,b,-)$ which in turn has the same sign as

$$S = \mu_1(b_2,\delta) - \mu_1(b,\delta).$$

Using the properties of $\mu_1(b,\delta)$ given in Lemma 5, S is positive if $b < b_2$ and negative if $b > b_2$. The result now follows.

4.8. Remarks and Exercises

Remark 1. Consider equation (4.1.1) under the hypothesis that the linearization is given by $\dot{x} = A(\varepsilon)x$ where $A(\varepsilon)$ is defined by (4.1.2). Suppose also that (4.2.1) and (H1) hold. Make the change of variables $\bar{x} = B(\varepsilon)x$, where $B(\varepsilon) = I - r\alpha^{-1}A(\varepsilon)$ and

$$r = \frac{\partial^3 f_1(0,0)}{\partial x_1^3}, \quad \alpha = \frac{\partial^3 f_2(0,0)}{\partial x_1^3}.$$

The map $x \to \bar{x}$ will be one-to-one for ε sufficiently small. Using the fact that $A(\varepsilon)$ and $B(\varepsilon)$ commute, it is easy to show that the transformed equation satisfies all the above hypotheses and that in addition it enjoys (H3). In particular, if the transformed equation is

$$\dot{\bar{x}} = F(\bar{x},\varepsilon)$$

then

$$\frac{\partial^3 F_1(0,0)}{\partial \bar{x}_1^3} = 0$$

$$\frac{\partial^3 F_2(0,0)}{\partial \bar{x}_1^3} = \frac{\partial^3 f_2(0,0)}{\partial x_1^3}$$

$$\frac{\partial^3 F_2(0,0)}{\partial \bar{x}_1^2 \partial \bar{x}_2} = \frac{\partial^3 f_2(0,0)}{\partial x_1^2 \partial x_2} + \frac{3\partial^3 f_1(0,0)}{\partial x_1^3} .$$

Remark 2. We have assumed that $f(x,\varepsilon) = -f(-x,\varepsilon)$. If we assume that this is true only for the low order terms then we would obtain similar results. For example, on L_1 we would get a homoclinic orbit with $x_1 > 0$. Similarly on another curve L_1' we would get a homoclinic orbit with $x_1 < 0$. In general L_1 and L_1' would be different although they would have the same linear approximation $\mu = (4/5)\delta$.

Remark 3. Suppose that we only assume (H1) and (H3). Then we can still obtain partial results about the local behavior of solutions. For example, in Lemma 2 we did not use the hypothesis β nonzero. Hence, for each $b > 0$, (4.1.1) has a periodic solution through $x_1 = 0$, $x_2 = |\alpha|^{-1/2}(2b)^{1/2}\varepsilon_1$ for some ε_1 and ε_2 with $\varepsilon_1 > 0$. However, we cannot say anything about the stability of the periodic orbits and we cannot say how many periodic orbits (4.1.1) has for fixed ε_1 and ε_2.

Exercises.

(1) Suppose that α and β are negative. Prove that for $(\varepsilon_1, \varepsilon_2)$ in regions 5 and 6, (4.1.1) has a connecting orbit. Hint: Use the calculations in Lemma 1 and Lemma 6.

(2) Suppose that α and β are negative and that $(\varepsilon_1, \varepsilon_2)$ is in region 3. Let U be the unstable manifold of the point $(0,0)$ in (4.1.1). Prove that U is in the region of attraction of the periodic orbit.

(3) Suppose that α is positive and β is negative. Show that the bifurcation set and the corresponding phase portraits are as given in Figures 3 and 5.

4.9. Quadratic Nonlinearities

In this section we discuss the local behavior of solutions of (4.1.1) when the nonlinearities are quadratic. Most of the material in this section can be found in [39].

Suppose that the linearization of (4.1.1) is $\dot{x} = A(\varepsilon)x$ where $A(\varepsilon)$ is given by (4.1.2) and that

$$f(0,\varepsilon) = 0, \quad \frac{\partial^2}{\partial x_1^2} f_1(0,0) = 0.$$

$$\alpha = \frac{\partial^2}{\partial x_1^2} f_2(0,0) \neq 0, \quad \beta = \frac{\partial^2}{\partial x_1 \partial x_2} f_2(0,0) \neq 0.$$

We also assume that $\beta > 0$, $\alpha < 0$; the results for the other cases are obtained by making use of the change of variables, $t \to -t$, $\varepsilon_2 \to -\varepsilon_2$, $x_1 \to \pm x_1$, $x_2 \to \mp x_2$.

Introduce parameters μ, δ, scaled variables y_1, y_2 and a new time $\bar{t}$ by the relations

$$\delta = |\varepsilon_1 \alpha^{-1}|^{1/2}, \quad x_1 = \delta^2 y_1, \quad x_2 = |\alpha|^{1/2}\delta^3 y_2,$$
$$\varepsilon_2 = |\alpha|^{1/2}\delta\mu, \quad t = \bar{t}|\alpha|^{-1/2}\delta^{-1}.$$

We asume that $\varepsilon_1 > 0$ in what follows, see Exercise 8 for the case $\varepsilon_1 < 0$.

After scaling, (4.1.1) becomes

$$\begin{aligned} \dot{y}_1 &= y_2 + O(\delta^2) \\ \dot{y}_2 &= y_1 + \mu y_2 - y_1^2 + \delta\gamma y_1 y_2 + O(\delta^2) \end{aligned} \tag{4.9.1}$$

where the dot means differentiation with respect to $\bar{t}$, the y_i lie in a bounded set and $\gamma = \beta|\alpha|^{-1/2}$. Note that by making a change of variable we can assume that $(1,0)$ is a fixed point of (4.9.1) for μ, δ sufficiently small.

The object of the following exercises is to show that the bifurcation set is given by Figure 6 and that the associated phase portraits are given by Figure 7.

Exercises.

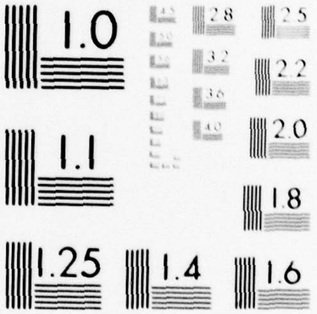
(4) Let $H(y_1, y_2) = (y_2^2/2) - (y_1^2/2) + (y_1^3/3)$. Show that along solutions of (4.9.1),

$$\dot{H} = \mu y_2^2 + \delta\gamma y_1 y_2^2 + O(\delta^2).$$

AD-A074 052 BROWN UNIV PROVIDENCE R I LEFSCHETZ CENTER FOR DYNAM--ETC F/G 12/1
APPLICATIONS OF CENTRE MANIFOLD THEORY.(U)
JUN 79 J CARR DAAG29-76-G-0294
UNCLASSIFIED ARO-13915.17-M NL

2 OF 2
AD
A074052

END
DATE
FILMED
10-79
DDC


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

(5) Prove that there is a function $\mu = \mu_1(c,\delta)$, $0 \le c \le 1$, such that if $\mu = \mu_1(c,\delta)$ in (4.9.1) then there is a periodic solution through $(c,0)$ if $0 < c < 1$ and a homoclinic orbit if $c = 0$.

(6) Prove that $\mu_1(c,\delta) = -\gamma P(c)\delta + O(\delta^2)$ where $J_0(c)P(c) = J_1(c)$,

$$J_i(c) = \int_c^{c_1} w^i R(w)dw,$$

$$R(w) = [w^2 - (2/3)w^3 + 2b]^{1/2},$$

$$6b = c^2(2c-3), \quad R(c_1) = 0, \quad c_1 > c.$$

(7) Prove that $P(0) = (6/7)$, $P(1) = 1$ and $P'(c) > 0$ for $0 < c < 1$. Deduce that for fixed μ,δ, (4.9.1) has at most one periodic solution. (To prove that $P'(c) > 0$ we can use the same techniques as in the proof of Lemma 4. An alternative method of proving $P'(c) > 0$ is given in [39].)

(8) If $\varepsilon_1 < 0$, then after scaling (4.1.1) becomes

$$\dot{y}_1 = y_2 + O(\delta^2)$$

$$\dot{y}_2 = -y_1 + \mu y_2 - y_1^2 + \gamma\delta y_1 y_2 + O(\delta^2).$$

Put $y_1 = z - 1$, $\bar{\mu} = \mu - \gamma\delta$. Then

$$\dot{z} = y_2 + O(\delta^2)$$
$$\dot{y}_2 = z + \bar{\mu}y_2 - z^2 + \delta\gamma z y_2 + O(\delta^2)$$

which has the same form as equation (4.9.1) and hence transform the results for $\varepsilon_1 > 0$ into results for the case $\varepsilon_1 < 0$.

(9) Show that the bifurcation set and the corresponding phase portraits are as given in Figures 6-7.

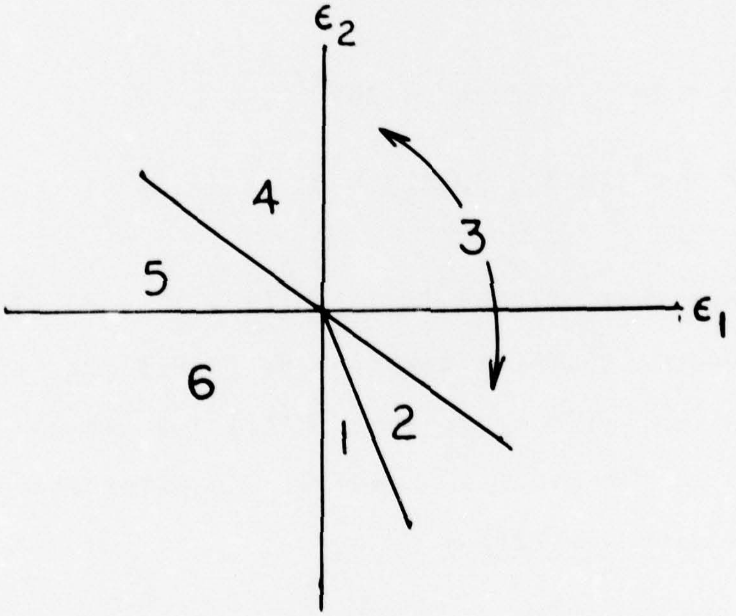


FIG. 6

Bifurcation Set for the Case $\alpha < 0$, $\beta > 0$.

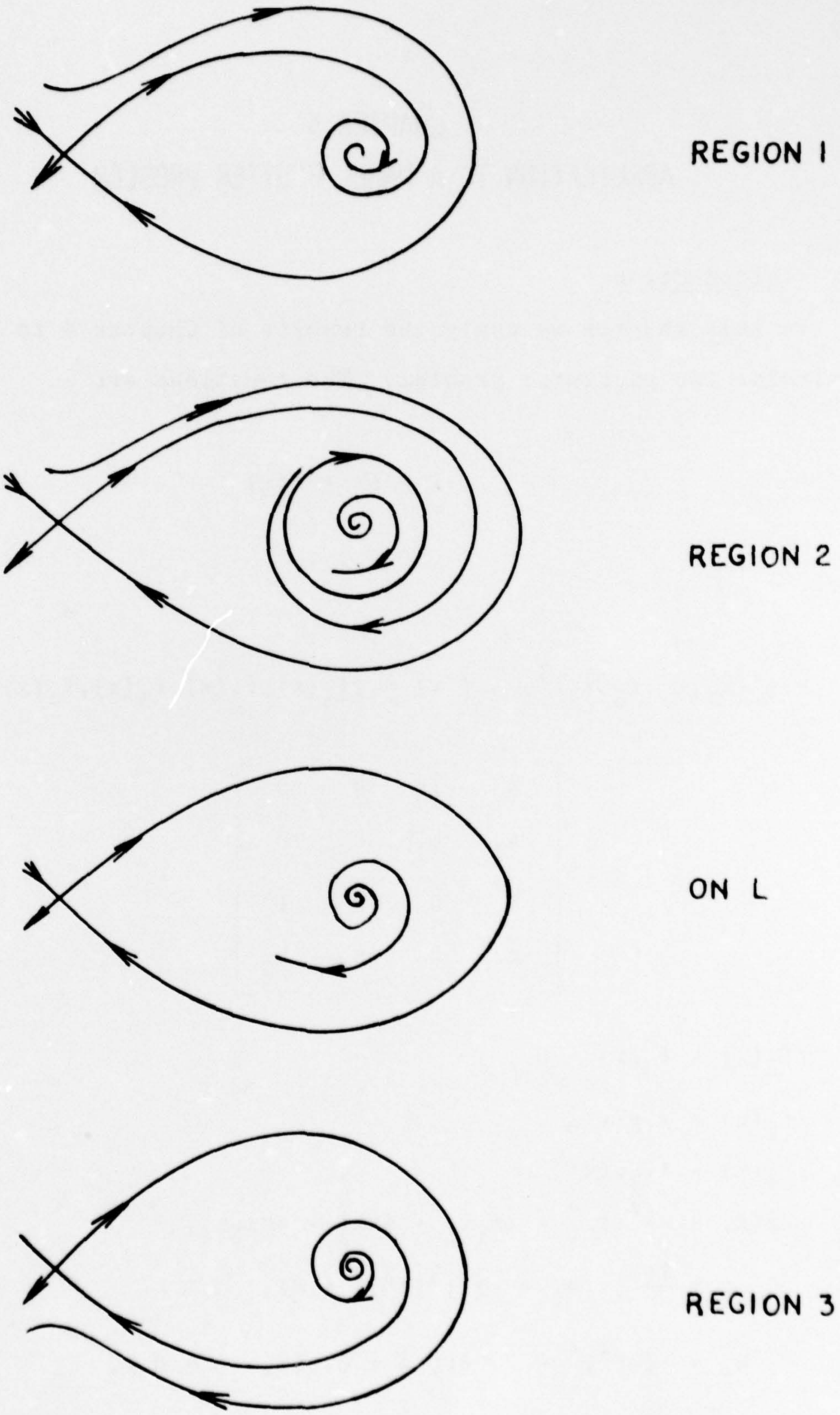

REGION 1
REGION 2
ON L
REGION 3

FIG. 7
Phase Portraits for the Case $\alpha < 0$, $\beta > 0$. (For the phase portraits in regions 4-6 use the transformations in Exercise 8.)

CHAPTER 5

APPLICATION TO A PANEL FLUTTER PROBLEM

5.1. Introduction

In this chapter we apply the results of Chapter 4 to a particular two parameter problem. The equations are

$$\dot{x} = Ax + f(x) \tag{5.1.1}$$

where

$$x = [x_1, x_2, x_3, x_4]^T, \quad f(x) = [f_1(x), f_2(x), f_3(x), f_4(x)]^T,$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_1 & b_1 & c & 0 \\ 0 & 0 & 0 & 1 \\ -c & 0 & a_2 & b_2 \end{bmatrix}$$

$$f_1(x) = f_3(x) = 0,$$

$$f_2(x) = x_1 g(x),$$

$$f_4(x) = 4x_3 g(x),$$

$$2g(x) = -\pi^4(kx_1^2 + \sigma x_1 x_2 + 4kx_3^2 + 4\sigma x_3 x_4),$$

$$c = \frac{8\rho}{3}, \quad a_j = -\pi^2 j^2 [\pi^2 j^2 + \Gamma],$$

$$b_j = -[\alpha\pi^4 j^4 + \sqrt{\rho}\,\delta]; \; \alpha = 0.005, \quad \delta = 0.1,$$

$k > 0$, $\sigma > 0$ are fixed and ρ, Γ are parameters. The above system results from a two mode approximation to a certain partial differential equation which describes the motion of a thin panel.

Holmes and Marsden [32,34] have studied the above equation and first we briefly describe their work. By numerical calculations, they find that for $\rho = \rho_0 \simeq 108$, $\Gamma = \Gamma_0 \simeq -2.23\pi^2$, the matrix A has two zero eigenvalues and two eigenvalues with negative real parts. Then for $|\rho-\rho_0|$ and $|\Gamma-\Gamma_0|$ small, by centre manifold theory, the local behavior of solutions of (5.1.1) is determined by a second order equation depending on two parameters. They then use some results of Takens [49] on generic models to conjecture that the local behavior of solutions of (5.1.1) for $|\rho-\rho_0|$ and $|\Gamma-\Gamma_0|$ small can be modelled by the equation

$$\ddot{u} + a\dot{u} + bu + u^2\dot{u} + u^3 = 0$$

for a and b small.

Recently, this conjecture has been proved by Holmes [33], in the case $\sigma = 0$, by reducing the equation on the centre manifold to Takens normal form. We use centre manifold theory and the results of Chapter 4 to obtain a similar result.

5.2. Reduction to a Second Order Equation

The eigenvalues of A are the roots of the equation

$$\lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4 = 0 \qquad (5.2.1)$$

where the d_i are functions of Γ and ρ. If A has two zero eigenvalues then $d_3 = d_4 = 0$. A calculation shows that if $d_3 = d_4 = 0$, then

$$\begin{aligned} a_1a_2 + c^2 &= 0 \\ a_1b_2 + b_1a_2 &= 0 \end{aligned} \qquad (5.2.2)$$

or in terms of Γ and ρ,

$$4\pi^2(\pi^2+\Gamma)(4\pi^2+\Gamma) + \frac{64}{9}\rho^2 = 0 \qquad (5.2.3)$$

$$(16\alpha\pi^4+\delta\rho^{1/2})(\pi^2+\Gamma) + 4(\alpha\pi^2+\delta\rho^{1/2})(4\pi^2+\Gamma) = 0. \qquad (5.2.4)$$

We prove that (5.2.3), (5.2.4) has a solution $\Gamma = \Gamma_0$, $\rho = \rho_0$.

From (5.2.3) we can express ρ in terms of Γ. Substituting this relation into (5.2.4) we obtain an equation $H(\Gamma) = 0$. Calculations show that $H(\Gamma_1) < 0$, $H(\Gamma_2) > 0$ where $\Gamma_1 = -(2.225)\pi^2$ and $\Gamma_2 = -(2.23)\pi^2$, so that $H(\Gamma_0) = 0$ for some $\Gamma_0 \in (\Gamma_2,\Gamma_1)$. Further calculations show that (5.2.3), (5.2.4) has a solution Γ_0,ρ_0 with $107.7 < \rho_0 < 107.8$.

In the subsequent analysis, we have to determine the sign of various functions of Γ_0 and ρ_0. Since we do not know Γ_0 and ρ_0 exactly we have to determine the sign of these functions for Γ_0 and ρ_0 in the above numerical ranges.

When $\Gamma = \Gamma_0$, $\rho = \rho_0$, the remaining eigenvalues of A are given by

$$\lambda_{3,4} = \frac{(b_1+b_2) \pm [(b_1-b_2)^2 + 4(a_1+a_2)]^{1/2}}{2},$$

and a calculation shows that they have negative real parts and non-zero imaginary parts.

We now find a basis for the appropriate eigenspaces when $\Gamma = \Gamma_0$, $\rho = \rho_0$. Solving $Av_1 = 0$ we find that

$$v_1 = [1,0,-a_1/c,0]^T. \tag{5.2.5}$$

The null space of A is in fact one-dimensional so the canonical form of A must contain a Jordan block. Solving $A^2v_2 = 0$ we obtain

$$v_2 = [0,1,-b_1/c,-a_1/c]^T, \quad Av_2 = v_1. \tag{5.2.6}$$

The vectors v_1 and v_2 form a basis for the generalized eigenspace of A corresponding to the zero eigenvalues. Similarly, we find a (real) basis for the space V spanned by the eigenvectors corresponding to λ_3 and λ_4. Solving $Az = \lambda_3 z$, we find that V is spanned by v_3 and v_4 where

$$\begin{aligned} 2v_3 &= z + \bar{z}, \quad 2v_4 = i(\bar{z}-z), \\ z &= [1,\lambda_3,w,\lambda_3 w]^r \\ wc &= b_2\lambda_3 - b_1b_2 + a_2. \end{aligned} \tag{5.2.7}$$

Let A_0 denote the matrix A when $\Gamma = \Gamma_0$ and $\rho = \rho_0$.

t $S = [v_1, v_2, v_3, v_4]$ where the v_i are defined by (5.2.5), 5.2.6) and (5.2.7) and set $y = S^{-1}x$. Then (5.1.1) can be written n the form

$$\dot{y} = By + S^{-1}(A - A_0)Sy + F(y,\Gamma,\rho) \tag{5.2.8}$$

where $F(y,\Gamma,\rho) = S^{-1}f(Sy)$,

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_1 & p_2 \\ 0 & 0 & -p_2 & p_1 \end{bmatrix}$$

and where $\lambda_3 = p_1 + ip_2$, $p_1 < 0$, $p_2 \neq 0$.

Then for $|\Gamma - \Gamma_0|$ and $|\rho - \rho_0|$ sufficiently small (5.2.8) has a centre manifold $y_3 = h_1(y_1,y_2,\Gamma,\rho)$, $y_4 = h_2(y_1,y_2,\Gamma,\rho)$. The flow on the centre manifold is governed by an equation of the form

$$\begin{bmatrix} \dot{y}_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + E(\Gamma,\rho) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + N(y_1,y_2,\Gamma,\rho) \tag{5.2.9}$$

where $E(\Gamma,\rho)$ is a 2×2 matrix with $E(\Gamma_0,\rho_0) = 0$ and $N(y_1,y_2,\Gamma,\rho)$ contains no linear terms in y_1 or y_2. We show that there is a nonsingular change of variables $(\Gamma,\rho) \to (\varepsilon_1,\varepsilon_2)$ for (Γ,ρ) close to (Γ_0,ρ_0) and a Γ,ρ dependent change of variables $(y_1,y_2) \to (\bar{y}_1,\bar{y}_2)$ such that the linearized equation

Let $S = [v_1, v_2, v_3, v_4]$ where the v_i are defined by (5.2.5), (5.2.6) and (5.2.7) and set $y = S^{-1}x$. Then (5.1.1) can be written in the form

$$\dot{y} = By + S^{-1}(A - A_0)Sy + F(y, \Gamma, \rho) \qquad (5.2.8)$$

where $F(y, \Gamma, \rho) = S^{-1}f(Sy)$,

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_1 & p_2 \\ 0 & 0 & -p_2 & p_1 \end{bmatrix}$$

and where $\lambda_3 = p_1 + ip_2$, $p_1 < 0$, $p_2 \neq 0$.

Then for $|\Gamma - \Gamma_0|$ and $|\rho - \rho_0|$ sufficiently small (5.2.8) has a centre manifold $y_3 = h_1(y_1, y_2, \Gamma, \rho)$, $y_4 = h_2(y_1, y_2, \Gamma, \rho)$. The flow on the centre manifold is governed by an equation of the form

$$\dot{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + E(\Gamma, \rho) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + N(y_1, y_2, \Gamma, \rho) \qquad (5.2.9)$$

where $E(\Gamma, \rho)$ is a 2×2 matrix with $E(\Gamma_0, \rho_0) = 0$ and $N(y_1, y_2, \Gamma, \rho)$ contains no linear terms in y_1 or y_2. We show that there is a nonsingular change of variables $(\Gamma, \rho) \to (\varepsilon_1, \varepsilon_2)$ for (Γ, ρ) close to (Γ_0, ρ_0) and a Γ, ρ dependent change of variables $(y_1, y_2) \to (\bar{y}_1, \bar{y}_2)$ such that the linearized equation corresponding to (5.2.9) is

$$\begin{bmatrix} \dot{\bar{y}}_1 \\ \bar{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} . \qquad (5.2.10)$$

The transformation $(y_1,y_2) \to (\bar{y}_1,\bar{y}_2)$ is of the form Identity + $O(|\Gamma-\Gamma_0|, |\rho-\rho_0|)$ and $\Gamma = \Gamma_0$, $\rho = \rho_0$ is mapped into $\varepsilon_1 = 0$, $\varepsilon_2 = 0$. After these transformations, (5.2.9) takes the form

$$\begin{bmatrix} \dot{y}_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + N(y_1,y_2,\Gamma_0,\rho_0) + \bar{N}(y_1,y_2,\Gamma,\rho) \qquad (5.2.11)$$

where we have dropped the bars on the y_i. The function $\bar{N}$ will contain no linear terms in y_1 or y_2 and $\bar{N}(y_1,y_2,\Gamma_0,\rho_0) = 0$. Since the nonlinearities in (5.1.1) are cubic, the same will be true of N and $\bar{N}$.

Let

$$N(y_1,y_2,\Gamma_0,\rho_0) = [N_1(y_1,y_2,\Gamma_0,\rho_0), N_2(y_1,y_2,\Gamma_0,\rho_0)]^T \qquad (5.2.12)$$

and let

$$\begin{aligned} \alpha_1 &= \frac{\partial^3}{\partial y_1^3} N_2(0,0,\Gamma_0,\rho_0) \\ r &= \frac{\partial^3}{\partial y_1^3} N_1(0,0,\Gamma_0,\rho_0) \\ \bar{\beta} &= \frac{\partial^3}{\partial y_1^2 \partial y_2} N_2(0,0,\Gamma_0,\rho_0) \\ \beta &= 3r + \bar{\beta}. \end{aligned} \qquad (5.2.13)$$

Using the results in Chapter 4 (see, in particular, Remark 1 in Section 8), if α_1 and β are non-zero, we can determine the local behavior of solutions of (5.2.11). By Theorem 2 of Chapter 2, this determines the local behavior of solutions of (5.2.8).

5.3. Calculation of the Linear Terms

From (5.2.1), $\mathrm{trace}(A) = d_3(\Gamma,\rho)$, $\det(A) = d_4(\Gamma,\rho)$ where

$$d_3(\Gamma,\rho) = 4\pi^2(\pi^2+\Gamma)(4\pi^2+\Gamma) + \frac{64}{9}\rho^2$$

$$d_4(\Gamma,\rho) = \pi^2(16\alpha\pi^4+\delta\rho^{1/2})(\pi^2+\Gamma) + 4\pi^2(\alpha\pi^2+\delta\rho^{1/2})(4\pi^2+\Gamma).$$

Calculations show that the mapping

$$(\Gamma,\rho) \to (d_3(\Gamma,\rho),d_4(\Gamma,\rho)) \qquad (5.3.1)$$

has non-zero Jacobian at $(\Gamma,\rho) = (\Gamma_0,\rho_0)$.

Define the matrix $C(\Gamma,\rho)$

$$C(\Gamma,\rho) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + E(\Gamma,\rho)$$

and let J be the value of the Jacobian of the mapping

$$(\Gamma,\rho) \to (\mathrm{trace}(C(\Gamma,\rho)),\ \det(C(\Gamma,\rho))), \qquad (5.3.2)$$

evaluated at (Γ_0,ρ_0). By considering the 4×4 matrix $B + S^{-1}(A-A_0)S$, it is easily seen that J is a non-zero multiple of the Jacobian of the mapping given by (5.3.1). Hence J is non-zero, so by the implicit function theorem we can use $\varepsilon_1 = -\det(C(\Gamma,\rho))$, $\varepsilon_2 = \mathrm{trace}(C(\Gamma,\rho))$ as our bifurcation parameters. Approximate formulae for ε_1 and ε_2 can easily be found if so desired.

Let $C(\Gamma,\rho) = [c_{ij}]$,

$$M = \begin{bmatrix} 1 & 0 \\ c_{11} & c_{12} \end{bmatrix}$$

$$\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = M \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} .$$

Then the linearized equation corresponding to (5.2.9) is

$$\begin{bmatrix} \dot{\bar{y}}_1 \\ \bar{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} .$$

Note that M is equal to the identity matrix when $\varepsilon_1 = \varepsilon_2 = 0$.

5.4. Calculation of the Nonlinear Terms

We now calculate the nonlinear term in (5.2.9) when $\Gamma = \Gamma_0$ $\rho = \rho_0$. Since the nonlinearities in (5.1.1) are cubic, the centre manifold has a "cubic zero" at the origin. Using $x = Sy$, on the centre manifold

$$x_1 = y_1 + O_3, \quad x_2 = y_2 + O_3$$
$$x_3 = \frac{-a_1}{c} y_1 - \frac{b_1}{c} y_2 + O_3, \quad x_4 = \frac{-a_1}{c} y_2 + O_3 \qquad (5.4.1)$$

where $O_3 = O(|y_1|^3 + |y_2|^3)$.

Let $S^{-1} = [t_{ij}]$ and let

$$F_j(y_1,y_2) = f_j(y_1,y_2,h_1(y_1,y_2,\Gamma_0,\rho_0)).$$

Then using the notation introduced in (5.2.12)

$$N_1(y_1,y_2,\Gamma_0,\rho_0) = t_{12}F_2(y_1,y_2) + t_{14}F_4(y_1,y_2)$$
$$N_2(y_1,y_2,\Gamma_0,\rho_0) = t_{22}F_2(y_1,y_2) + t_{24}F_4(y_1,y_2). \qquad (5.4.2)$$

Using (5.4.1)

$$F_2(y_1,y_2) = \frac{-\pi^4}{2}\left[ky_1^3+4k\left(\frac{a_1}{c}\right)^2 y_1^3 + \sigma y_1^2 y_2 + \frac{8ka_1b_1}{c^2} y_1^2 y_2 + \frac{4\sigma a_1^2}{c^2} y_1^2 y_2\right]$$
$$+ \text{ terms in } y_1 y_2^2 + O_5$$

$$F_4(y_1,y_2) = \frac{-4a_1}{c} F_2(y_1,y_2) + \frac{2b_1\pi^4}{c}\left[ky_1^2 y_2 + 4k\left(\frac{a_1}{c}\right)^2 y_1^2 y_2\right]$$
$$+ \text{ terms in } y_1 y_2^2 \text{ and } y_2^3 + O_5$$

where $O_5 = O(|y_1|^5 + |y_2|^5)$. Note also that since $\Gamma = \Gamma_0$ and $\rho = \rho_0$, (5.2.2) holds.

From (5.2.13) and (5.4.2)

$$\alpha_1 = \frac{-\pi^4}{2} k(1 + 4(\frac{a_1}{c})^2)(t_{22} - 4\frac{a_1}{c} t_{24})$$

$$r = \frac{-\pi^4}{2} k(1 + 4(\frac{a_1}{c})^2)(t_{12} - 4\frac{a_1}{c} t_{14})$$

$$\overline{\beta} = \frac{-\pi^4}{2} (\sigma + \frac{8ka_1b_1}{c^2} + \frac{4\sigma a_1^2}{c^2})(t_{22} - 4\frac{a_1}{c} t_{24}) + \frac{2\pi^4}{c^3} b_1 k t_{24}(c^2+4a_1^2).$$

Routine calculations show that

$$t_{12} = m^2 b_2(2a_1 - b_1 b_2)$$

$$t_{22} = ma_2$$

$$t_{14} = m^2 c(b_1 + b_2)$$

$$t_{24} = -mc$$

$$m = (a_1 + a_2 - b_1 b_2)^{-1} < 0.$$

Using (5.2.2) and numerical calculations, we find that

$$t_{22} - 4\frac{a_1}{c} t_{24} = m(4a_1 + a_2) > 0$$

$$t_{12} - 4\frac{a_1}{c} t_{14} = b_1 m^2(2a_2 - b_2^2 - 4a_1) > 0$$

so that α_1 and r are negative. Similarly, the coefficient of k in $\overline{\beta}$ is

$$\frac{-2\pi^4}{c^2} a_1 b_1 m(12a_1 + a_2) < 0.$$

Hence α_1 and β are negative and the local behavior of solutions of (5.1.1) can be determined using the results of Chapter 4.

CHAPTER 6

INFINITE DIMENSIONAL PROBLEMS

6.1. Introduction

In this chapter we extend centre manifold theory to a class of infinite dimensional problems. For simplicity we only consider equations of the form

$$\dot{w} = Cw + N(w), \quad w(0) \in Z,$$

where Z is a Banach space, C is the generator of a strongly continuous semigroup on Z and $N: Z \to Z$ is smooth. (For a discussion of semigroup theory see [4, 44].) With appropriate spectral assumptions on C, it is very easy to prove the analogues of Theorems 1-3 of Chapter 2. For generalizations to other evolution systems see [30,42].

6.2. Centre Manifolds

Let Z be a Banach space with norm $||\cdot||$. We consider ordinary differential equations of the form

$$\dot{w} = Cw + N(w), \quad w(0) \in Z, \qquad (6.2.1)$$

where C is the generator of a strongly continuous linear semigroup $S(t)$ and $N: Z \to Z$ has a uniformly continuous second derivative with $N(0) = 0$, $N'(0) = 0$ [N' is the Frechet derivative of N]. It can be shown [4,7] that, with an appropriate definition of weak solution, a function $w \in C([0,T];Z)$, $T > 0$, is a weak solution of (6.2.1) if and only if w satisfies

$$w(t) = S(t)w(0) + \int_0^t S(t-\tau)N(w(\tau))d\tau \tag{6.2.2}$$

on $[0,T]$. A standard argument shows that there is a unique solution of (6.2.2) defined on some maximal interval $[0,T_m)$, $T_m > 0$, and that if $T_m < \infty$ then

$$\lim_{t \to T_m^-} ||w(t)|| = \infty.$$

Furthermore, if $w(0) \in D(A)$ then $w(t)$ is differentiable on $(0,T_m)$ and satisfies (6.2.1).

To put (6.2.1) into canonical form we make some spectral assumptions about C. We assume from now on that:

(i) $Z = X \oplus Y$ where X is finite dimensional and Y is closed.

(ii) X is C-invariant and that if A is the restriction of C to X, then the real parts of the eigenvalues of A are all zero.

(iii) If $U(t)$ is the restriction of $S(t)$ to Y, then Y is $U(t)$-invariant and for some positive constants a,b,

$$||U(t)|| \leq ae^{-bt}, \quad t \geq 0. \tag{6.2.3}$$

Let P be the projection on X along Y. Let $B = (I-P)C$ and for $x \in X$, $y \in Y$, let $f(x,y) = PN(x+y)$, $g(x,y) = (I-P)N(x+y)$.

Equation (6.2.1) can be written

$$\dot{x} = Ax + f(x,y)$$
$$\dot{y} = By + g(x,y) . \qquad (6.2.4)$$

An invariant manifold for (6.2.4) which is tangent to X space at the origin is called a centre manifold.

Theorem 1. There exists a centre manifold for (6.2.4), $y = h(x)$, $|x| < \delta$ where h is C^2.

The proof of Theorem 1 is exactly the same as the proof given in Chapter 2 for the corresponding finite dimensional problem.

The equation on the centre manifold is given by

$$\dot{u} = Au + f(u,h(u)). \qquad (6.2.5)$$

In general if $y(0)$ is not in the domain of B then $y(t)$ will not be differentiable. However, on the centre manifold $y(t) = h(x(t))$, and since X is finite dimensional $x(t)$, and consequently $y(t)$, are differentiable.

Theorem 2 (a) Suppose that the zero solution of (6.2.5) is stable (asymptotically stable) (unstable). Then the zero solution of (6.2.4) is stable (asymptotically stable) (unstable).

(b) Suppose that the zero solution of (6.2.5) is stable Let $(x(t),y(t))$ be a solution of (6.2.4) with $||(x(0),y(0)||$

sufficiently small. Then there exists a solution $u(t)$ of (6.2.5) such that as $t \to \infty$,

$$\begin{aligned} x(t) &= u(t) + O(e^{-\gamma t}) \\ y(t) &= h(u(t)) + O(e^{-\gamma t}) \end{aligned} \tag{6.2.6}$$

where $\gamma > 0$.

The proof of the above theorem is exactly the same as the proof given for the corresponding finite dimensional result.

Using the invariance of h and proceeding formally we have that

$$h'(x)[Ax + f(x,h(x))] = Bh(x) + G(x,h(x)). \tag{6.2.7}$$

To prove that equation (6.2.7) holds we must show that $h(x)$ is in the domain of B.

Let $x_0 \in X$ be small. To prove that $h(x_0)$ is in the domain of B it is sufficient to prove that

$$\lim_{t\to 0^+} \frac{U(t)h(x_0) - h(x_0)}{t}$$

exists. Let $x(t)$, $y(t) = h(x(t))$ be the solution of (6.2.4) with $x(0) = x_0$. As we remarked earlier, $y(t)$ is differentiable. From (6.2.4)

$$y(t) = U(t)h(x_0) + \int_0^t U(t-\tau)g(x(\tau),y(\tau))d\tau,$$

so it is sufficient to prove that

$$\lim_{t\to 0^+} \frac{1}{t} \int_0^t u(t-\tau) g(x(\tau), y(\tau)) d\tau$$

exists. This easily follows from the fact that $U(t)$ is a strongly continuous semigroup and g is smooth. Hence $h(x_0)$ is in the domain of B.

Theorem 3. Let ϕ be a C^1 map from a neighborhood of the origin in X into Y such that $\phi(0) = 0$, $\phi'(0) = 0$ and $\phi(x) \in D(B)$. Suppose that as $x \to 0$, $(M\phi)(x) = O(|x|^q)$, $q > 1$, where

$$(M\phi)(x) = \phi'(x)[Ax + f(x,\phi)] - B\phi(x) - g(x,\phi(x)).$$

Then as $x \to 0$, $||h(x) - \phi(x)|| = O(|x|^q)$.

The proof of Theorem 3 is the same as that given for the finite dimensional case except that the extension $\theta : X \to Y$ of ϕ must be defined so that $\theta(x)$ is in the domain of B.

6.3. Examples

Example 1. Consider the semilinear wave equation

$$\begin{gathered} v_{tt} + v_t - v_{xx} - v + f(v) = 0, \ (x,t) \in (0,\pi) \times (0,\infty) \\ v = 0 \ \text{ at } \ x = 0,\pi \end{gathered} \tag{6.3.1}$$

where f is a C^3 function satisfying $f(v) = v^3 + O(v^4)$ as $v \to 0$. We first formulate (6.3.1) as an equation on a Hilbert space

Let $Q = (d/dx)^2 + 1$, $D(Q) = H^2(0,1) \cap H_0^1(0,1)$. Then Q is a self-adjoint operator. Let $Z = H_0^1(0,1) \times L^2(0,1)$, then (6.3.1) can be rewritten as

$$\dot{w} = Cw + N(w) \tag{6.3.2}$$

where

$$Cw = \begin{bmatrix} w_2 \\ Qw_1 - w_2 \end{bmatrix}, \quad N(w) = \begin{bmatrix} 0 \\ -f(w_1) \end{bmatrix}.$$

Since C is the sum of a skew self-adjoint operator and a bounded operator, C generates a strongly continuous group. Clearly N is a C^3 map from Z into Z.

The eigenvalues of C are $\lambda_n = [-1 \pm (5-4n^2)^{1/2}]/2$. $\lambda_1^+ = 0$ and all the other eigenvalues have real part less than 0. The eigenspace corresponding to the zero eigenvalue is spanned by q_1 where

$$q_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin x.$$

To apply the theory of Section 2, we must put (6.3.2) into canonical form. We first note that $Cq_2 = -q_2$ where

$$q_2(x) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin x$$

and that all the other eigenspaces are spanned by elements of the form $\alpha_n \sin nx$, $n \geq 2$, $\alpha_n \in \mathbb{R}^2$. In particular, all other eigenvectors are orthogonal to q_1 and q_2. Let $X = \operatorname{span}(q_1)$, $V = \operatorname{span}(q_1, q_2)$, $Y = \operatorname{span}(q_2) \oplus V^{\perp}$, then $Z = X \oplus Y$. The projection $P: Z \to X$ is given by

$$P\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = (\bar{w}_1 + \bar{w}_2)q_1 \tag{6.3.3}$$

where

$$\bar{w}_j(x) = \frac{2}{\pi}\int_0^{\pi} w_j(\theta)\sin\theta \, d\theta.$$

Let $w = sq_1 + y$, $s \in \mathbb{R}$, $y \in Y$ and $B = (I-P)C$, then we can write (6.3.2) in the form

$$\begin{aligned} \dot{s}q_1 &= PN(sq_1+y) \\ \dot{y} &= By + (I-P)N(sq_1+y). \end{aligned} \tag{6.3.4}$$

By Theorem 1, (6.3.4) has a centre manifold $y = h(s)$, $h(0) = 0$, $h'(0) = 0$, $h: (-\delta, \delta) \to Y$. By Theorem 2, the equation which determines the asymptotic behavior of solutions of (6.3.4) is the one-dimensional equation

$$sq_1 = PN(sq_1+h(s)). \qquad (6.3.5)$$

Since the nonlinearities in (6.3.4) are cubic, $h(s) = O(s^3)$, so that

$$\dot{s} = -\frac{2}{\pi}\int_0^{\pi} f(s+O(s^3))\sin^4\theta \, d\theta$$

or

$$\dot{s} = -\frac{3}{4} s^3 + O(s^4). \qquad (6.3.6)$$

Hence, by Theorem 2, the zero solution of (6.3.6) is asymptotically stable. Using the same calculations as in Section 1 of Chapter 3, if $s(0) > 0$ then as $t \to \infty$,

$$s(t) = (\tfrac{3}{2} t)^{-1/2} + O(t^{-1/2}). \qquad (6.3.7)$$

Hence, if $v(x,t)$ is a solution of (6.3.1) with $v(x,0)$, $v_t(x,0)$ small, then either $v(x,t)$ tends to zero exponentially fast or

$$v(x,t) = \pm s(t)\sin x + O(s^3) \qquad (6.3.8)$$

where $s(t)$ is given by (6.3.7).

Further terms in the above asymptotic expansion can be calculated if we have more information about f. Suppose that $f(v) = v^3 + av^5 + O(v^7)$ as $v \to 0$. In order to calculate an approximation to $h(s)$ set

$$(M\phi)(s) = \phi'(s)PN(sq_1+\phi(x)) - B\phi(s) - (I-P)N(sq_1+\phi(s)) \quad (6.3.9)$$

where $\phi: \mathbb{R} \to Y$. To apply Theorem 3 we choose $\phi(s)$ so that $(M\phi)(s) = O(s^5)$. If $\phi(s) = O(s^3)$ then

$$\begin{aligned} M\phi(s) &= -B\phi(s) - (I-P)N(sq_1) + O(s^5) \\ &= -B\phi(s) - \frac{3}{4}s^3q_2 - \frac{s^3}{4}\begin{bmatrix}0\\1\end{bmatrix}q + O(s^5) \end{aligned} \quad (6.3.10)$$

where $q(x) = \sin 3x$. If

$$\phi(s) = \alpha q_2 s^3 + \begin{bmatrix}\beta_1\\\beta_2\end{bmatrix}qs^3 \quad (6.3.11)$$

then substituting (6.3.11) into (6.3.10) we obtain

$$(M\phi)(s) = \alpha q_2 s^3 + \begin{bmatrix}-\beta_2\\8\beta_1+\beta_2\end{bmatrix}qs^3 - \frac{3}{4}s^3q_2 - \frac{s^3}{4}\begin{bmatrix}0\\1\end{bmatrix}q.$$

Hence, if $\alpha = 3/4$, $\beta_1 = 1/32$, $\beta_2 = 0$ then $M\phi(s) = O(s^5)$ so by Theorem 3

$$h(s) = \frac{3}{4}q_2s^3 + \frac{1}{32}\begin{bmatrix}1\\0\end{bmatrix}qs^3 + O(s^5). \quad (6.3.12)$$

Substituting (6.3.12) into (6.3.5) we obtain

$$\dot{s} = \frac{-3}{4}s^3 - \left(\frac{213}{128} + \frac{5}{8}a\right)s^5 + O(s^7).$$

The asymptotic behavior of solutions can now be found using the calculations given in Section 1 of Chapter 3.

Example 2. In this example we apply our theory to the equation

$$v_{tt} + v_t + v_{xxxx} - [\beta + (2/\pi^4)\int_0^1 (v_s(s,t))^2 ds]v_{xx} = 0, \quad (6.3.13)$$

with $v = v_{xx} = 0$ at $x = 0,1$ and given initial conditions $v(x,0)$, $v_t(x,0)$, $0 \le x \le 1$. Equation (6.3.13) is a model for the transverse motion of an elastic rod with hinged ends, v being the transverse deflection and β a constant. The above equation has been studied by Ball [5,6] and in particular he showed that when $\beta = -\pi^2$ the zero solution of (6.3.13) is asymptotically stable. However, in this case the linearized equations have a zero eigenvalue and so the rate of decay of solutions depends on the non-linear terms. In [11] the rate of decay of solutions of (6.3.13) was found using centre manifold theory. Here we discuss the behavior of small solutions of (6.3.13) when $\beta + \pi^2$ is small.

As in the previous example we formulate (6.3.13) as an ordinary differential equation. We write $'$ for space derivatives and $\cdot$ for time derivatives. Let

$$Z = H^2(0,1) \cap H_0^1(0,1) \times L^2(0,1), \quad Cv = -v'''' - \beta v'',$$

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad Cw = \begin{bmatrix} w_2 \\ Cw_1 - w_2 \end{bmatrix}, \quad N(w) = \begin{bmatrix} 0 \\ f(w_1) \end{bmatrix}$$

$$f(w_1) = \frac{2w_1''}{\pi^4}\int_0^1 w_1'(s)^2 ds.$$

Then we can write (6.3.13) as

$$\dot{w} = Cw + N(w). \tag{6.3.14}$$

It is easy to check that C generates a strongly continuous group on Z and that N is a C^∞ map from Z into Z.

If λ is an eigenvalue of C then we must have a nontrivial solution $u(x)$ of

$$u'''' - \beta u'' + (\lambda+\lambda^2)u = 0$$
$$u(0) = u''(0) = u(1) = u''(1) = 0.$$

An easy computation then shows that λ is an eigenvalue of C if and only if

$$2\lambda = -1 \pm [1 - 4(n^4\pi^4+\beta n^2\pi^2)]^{1/2}.$$

Let $\varepsilon = \pi^2\beta + \pi^4$. Then the eigenvalues of C are $\lambda_n(\varepsilon)$, where $2\lambda_1(\varepsilon) = -1 + [1-4\varepsilon]^{1/2}$, $\lambda_2(\varepsilon) = -1 - \lambda_1(\varepsilon)$, and all the rest of the $\lambda_n(\varepsilon)$ have real parts less than zero for ε sufficiently small. The eigenspaces corresponding to $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ are spanned by q_1 and q_2 where

$$q_1 = \begin{bmatrix} 1 \\ \lambda_1(\varepsilon) \end{bmatrix} \sin \pi x, \qquad q_2 = \begin{bmatrix} 1 \\ \lambda_2(\varepsilon) \end{bmatrix} \sin \pi x$$

while the eigenspaces corresponding to $\lambda_n(\varepsilon)$ for $n \geq 2$ are spanned by elements orthogonal to q_1 and q_2.

Let $X = \mathrm{span}(q_1)$, $V = \mathrm{span}(q_1, q_2)$, $Y = \mathrm{span}(q_2) \oplus V^{\perp}$, then $Z = X \oplus Y$ and the projection $P: Z \to X$ is given by

$$P\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = (\lambda_1(\varepsilon) - \lambda_2(\varepsilon))^{-1}(\bar{w}_2 - \lambda_2(\varepsilon)\bar{w}_1)q_1$$

where

$$\bar{w}_j = 2\int_0^1 w_j(\theta)\sin \pi\theta \, d\theta .$$

Let $w = sq_1 + y$, where $s \in \mathbb{R}$ and $y \in Y$. Then we can write (6.3.14) in the form

$$\begin{aligned} \dot{s}q_1 &= \lambda_1(\varepsilon)sq_1 + PN(sq_1+y) \\ \dot{y} &= By + (I-P)N(sq_1+y) \qquad (6.3.15) \\ \dot{\varepsilon} &= 0 . \end{aligned}$$

By Theorem 1, (6.3.15) has a centre manifold $y = h(s,\varepsilon)$, $|s| < \delta$, $|\varepsilon| < \varepsilon_0$. Using $h(s,\varepsilon) = O(s^2 + |\varepsilon s|)$, if $N_2(w)$ is the second component of $N(w)$ then

$$\begin{aligned} N_2(sq_1 + h(s,\varepsilon))(x) &= \frac{-2}{\pi^4}\left[\int_0^1 s^2\pi^4\cos^2\pi\theta \, d\theta\right]\pi^2 s \sin \pi x \\ &\quad + O(s^4 + |\varepsilon s^3|) \\ &= -s^3 \sin \pi x + O(s^4 + |\varepsilon s^3|). \end{aligned}$$

Hence $PN(sq_1 + h(s,\varepsilon)) = (-s^3 + O(s^4 + |\varepsilon s^3|))q_1$, so that by Theorem 2, the asymptotic behavior of small solutions of (6.3.15) is determined by the equation

$$\dot{s} = \lambda_1(\varepsilon)s - s^3 + O(|\varepsilon s^3| + |s^4|) \qquad (6.3.16)$$

We can now determine the asymptotic behavior of small solutions of (6.3.14). For $0 < \varepsilon < \delta$, solutions of (6.3.14) are asymptotically stable. For $-\delta < \varepsilon < 0$, the unstable manifold of the origin consists of two stable orbits connecting two fixed points to the origin. (See Figure 1)

Problem. For $\varepsilon = 0$, show that equation (6.3.16) can be written as

$$\dot{s} = -s^3 - 3s^5 + O(s^7)$$

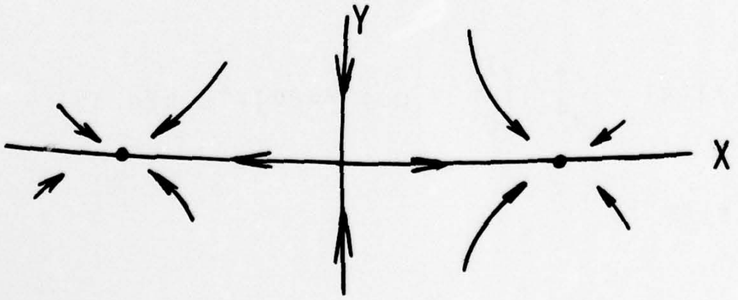


Figure 1
Phase Portrait for Small Negative ε

Example 3. Consider the equation

$$
\begin{aligned}
&u_{tt} + 2u_t - u_{xx} + a^2 v + f(u,v) = 0 \\
&v_{tt} + 2v_t - v_{xx} - u + g(u,v) = 0
\end{aligned}
\qquad (6.3.17)
$$

for $(x,t) \in (0,\pi) \times (0,\infty)$ with $u = v = 0$ at $x = 0,\pi$, where $f(u,v)$, $g(u,v)$ have a second order zero at $u = v = 0$. For $a = 2$, we show that the linearized problem has two purely imaginary eigenvalues while all the rest have negative parts. We then use centre manifold theory to reduce the prbblem of bifurcation of periodic solutions to a two-dimensional problem.

Let $w = (u,v,\dot{u},\dot{v})^T$, then we can write (6.3 17) as

$$
w = Cw + N(w) \qquad (6.3.18)
$$

on $Z = (H^2(0,1) \cap H_0^1(0,1))^2 \times (L^2(0,1))^2$. Let

$$
A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u_{xx} + a^2 v \\ -v_{xx} + u \end{pmatrix}.
$$

If μ is an eigenvalue of A then λ is an eigenvalue of C where $\lambda^2 + 2\lambda + \mu = 0$ and all the eigenvalues of C arise in this way. An easy calculation shows that the eigenvalues of C are given by

$$
\lambda = -1 \pm (1-n^2 \pm ia)^{1/2}, \quad n = 1,2,\ldots .
$$

For $a = 2$, the eigenvalues of C are $\lambda_1 = i$, $\overline{\lambda}_1 = -i$, while all the rest have negative real parts. Also

$$\frac{d}{da} (\text{Re } \lambda_1(2)) > 0$$

so that $\lambda_1(a)$ and $\overline{\lambda}_1(a)$ cross the imaginary axis with non-zero speed. It is now trivial to apply centre manifold theory to conclude that for $a - 2$ small, the behavior of small solutions is determined by an equation of the form

$$\dot{s} = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} s + J(s,\alpha) \qquad (6.3.19)$$

where $s \in \mathbb{R}^2$, α is a real parameter and $J(s,\alpha) = O(s^2)$. To apply the theory in Section 2, Chapter 3, we need to calculate the quadratic and cubic terms in $J(s,0)$. To do this we need to put (6.3.18) into canonical form and to calculate the centre manifold when $a = 2$. From now on we let $a = 2$.

On the subspace $\{r \sin nx: r \in \mathbb{R}^4\}$ the operator C can be represented by the matrix C_n where

$$C_n = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -n^2 & -4 & -2 & 0 \\ 1 & -n^2 & 0 & -2 \end{bmatrix} .$$

Note that the eigenvalues of C are given by the eigenvalues of C_n for $n = 1,2,\ldots$. To put (6.3.18) into canonical form we first find a basis which puts C_1 into canonical form. Calculations show that if

$$q_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad q_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} \quad q_3 = \begin{bmatrix} 0 \\ -1 \\ -2 \\ 2 \end{bmatrix} \quad q_4 = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

then $C_1q_1 = -q_2$, $C_1q_2 = q_1$, $C_1q_3 = -2q_3 + q_4$, $C_1q_4 = -q_3 - 2q_4$. Let $w = Qz$ where $Q = [q_1,q_2,q_3,q_4]$, then we can rewrite (6.3.18) as

$$z = Q^{-1}CQz + Q^{-1}N(Qw). \tag{6.3.20}$$

Let $X = \{[s_1,s_2,0,0]^T\psi : s_1,s_2 \in \mathbb{R}\}$, $V = \{[0,0,r_1,r_2]^T\psi : r_1,r_2 \in \mathbb{R}\}$, $Y = V \oplus [X \oplus V]^\perp$ where $\psi(x) = \sin x$. Then $Z = X \oplus Y$ and the projection P on X along Y is given by

$$Pw = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ 0 \\ 0 \end{bmatrix} \psi$$

$$\bar{w}_j(x) = \frac{2}{\pi}\int_0^\pi w_j(\theta)\sin\theta\, d\theta.$$

Let $z = [s_1,s_2,0,0]^T\psi + y$, $s_i \in \mathbb{R}$, $y \in Y$, then we can write (6.3.20) in the form

$$\begin{aligned} \dot{s} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} s + \langle PQ^{-1}N(Qw), \phi \rangle \\ \dot{y} &= By + (I-P)Q^{-1}N(Qw) \end{aligned} \tag{6.3.21}$$

where $B = (I-P)Q^{-1}CQ$, $\phi = [1,1,0,0]^T\psi$ and $s = [s_1, s_2]^T$. By Theorem 1 (6.3.21) has a centre manifold $y = h(s)$.

From $w = Qz$ we have that $w_1 = -2z_1 - 2z_4$ and $w_2 = z_2 - z_3$. Let $Q^{-1} = (t_{ij})$,

$$\begin{bmatrix} F(z) \\ G(z) \end{bmatrix} = \begin{bmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \end{bmatrix} \begin{bmatrix} f(-2z_1 - 2z_4, z_2 - z_3) \\ g(-2z_1 - 2z_4, z_2 - z_3) \end{bmatrix} .$$

Suppose that $F(z) = F_3(z) + O(|z|^4)$ and $G(z) = G_3(z) + O(|z|^4)$ where F_3 and F_4 are homogeneous cubics. Then if $J_1(s)$, $J_2(s)$ denote the first two components of $\langle PQ^{-1}N(Qw), \phi \rangle$ on the centre manifold,

$$J_1(s) = \frac{2}{\pi} \int_0^{\pi} F_3(s_1 \sin\theta, s_2 \sin\theta, 0, 0)\, d\theta + O(|s|^4)$$

with a similar expression for $J_2(s)$. Hence, on the centre manifold

$$\dot{s} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} J_1(s) \\ J_2(s) \end{bmatrix} \tag{6.3.22}$$

and we can apply the theory given in Section 2 of Chapter 3. If the constant K associated with (6.3.22) is zero (see Section 2 of

Chapter 3 for the definition of K) then the above procedure gives no information and we have to calculate higher order terms.

If $F(z) = F_2(z) + F_3(z) + O(|z|^4)$ and $G(z) = G_2(z) + G_3(z) + O(|z|^4)$ where F_2 and G_2 are homogeneous quadratics then the calculation of the nonlinear terms is much more complicated. On the centre manifold $z_1 = s_1\psi + O(s^2)$, $z_2 = s_2\psi + O(s^2)$, $z_3 = O(s^2)$, and $z_4 = O(s^2)$. The terms of order s^2 make a contribution to the cubic terms in $PQ^{-1}N(Qw)$. Hence, we need to find a quadratic approximation to $h(s)$. This is straightforward but rather complicated so we omit the details.

Example 4. Consider the equations

$$\begin{aligned} u_t &= Du_{xx} + (B-1)u + A^2v + 2Auv + u^2v + BA^{-1}u^2 \\ v_t &= \theta Dv_{xx} - Bu - A^2v - 2Auv - u^2v - BA^{-1}u^2 \\ u &= v = 0 \quad \text{at} \quad x = 0,1, \end{aligned} \tag{6.3.23}$$

where A, B, θ, D are positive. The above equations come from a simplified model of a chemical reaction with $u + A$ and $v + BA^{-1}$ as the chemical concentrations [3,10].

We study (6.3.23) on $Z = (H_0^1(0,\pi))^2$. Set

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad C = \begin{bmatrix} D \dfrac{d^2}{dx^2} + (B-1) & A^2 \\ -B & \theta D \dfrac{d^2}{dx^2} - A^2 \end{bmatrix}$$

$$N(w) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (2Auv + u^2v + BA^{-1}u^2)$$

then we can write (6.3.23) as

$$\dot{w} = Cw + N(w). \tag{6.3.24}$$

We analyze the situation in which for some value of the parameters A,B,θ,D, C has two zero eigenvalues such that the restriction of C to the zero eigenspace has a Jordan block. The bifurcation of static solutions when C has two zero eigenvalues and the restriction of C to the zero eigenspace is zero, has been studied in [21,22].

On the subspace $\{r \sin n\pi x,\ r \in \mathbb{R}^2\}$ the operator C can be represented by the matrix

$$C_n = \begin{bmatrix} -n^2\pi^2 D + B - 1 & A^2 \\ -B & -\theta D n^2\pi^2 - A^2 \end{bmatrix}.$$

The eigenvalues of C are given by the eigenvalues of C_n for $n = 1,2,\ldots$. We suppose that two of the eigenvalues of C are negative while all the rest have negative real parts. For simplicity we assume that the eigenvalues of C_1 are zero.

If C_1 is to have two zero eigenvalues then

$$\begin{aligned} \operatorname{trace}(C_1) &= B - 1 - \pi^2 D - A^2 - \theta\pi^2 D = 0 \\ \det(C_1) &= A^2 B - (B-1-\pi^2 D)(A^2+\theta\pi^2 D) = 0. \end{aligned} \tag{6.3.25}$$

We make the following hypotheses:

(H1) There exists A_0, B_0, θ_0, D_0 such that (6.3.25) is satisfied and the real parts of the rest of the eigenvalues of C are negative.

(H2) For A, B, θ, D in a neighborhood of A_0, B_0, θ_0, D_0, we can parametrize $\operatorname{trace}(C_1)$ and $\det(C_1)$ by

$$\operatorname{trace}(C_1) = \varepsilon_2, \quad \det(C_1) = -\varepsilon_1. \tag{6.3.26}$$

The first hypothesis is satisfied, for example, if $D_0 > 0$ and

$$\theta_0 = 1, \quad A_0 = \pi^2 D_0, \quad B_0 = (1+\pi^2 D_0)^2. \tag{6.3.27}$$

If we vary B and θ and keep $A = A_0$, $D = D_0$, then the mapping $(B,\theta) \to (\det(C_1), \operatorname{trace}(C_1))$ has a non-zero Jacobian at $B = B_0$, $\theta = \theta_0$ if A_0, B_0, θ_0 are given by (6.3.27), so by the implicit function theorem, (H2) is satisfied. In order to simplify calculations we assume (6.3.27) from now on.

Let $X = \{s\psi : s \in \mathbb{R}^2\}$, $Y = X^{\perp}$ where $\psi(x) = \sin \pi x$, then $Z = X \oplus Y$. By Theorem 1, the system

$$\begin{aligned} \dot{w} &= Cw + N(w) \\ \dot{\varepsilon} &= 0 \end{aligned}$$

has a centre manifold h: (neighborhood of $X \times \mathbb{R}^2$) $\to Y$, where we have written $\varepsilon = (\varepsilon_1, \varepsilon_2)$. On the centre manifold, the equation reduces to

$$\dot{s} = C_1(\varepsilon)s + \overline{N}(s\psi + h(s,\varepsilon)) \tag{6.3.28}$$

where $s = [s_1, s_2]^T$ and

$$\overline{N}_i(z) = 2\int_0^1 N_i(z(\theta))\sin \pi\theta\, d\theta.$$

We treat the linear and nonlinear parts of (6.3.28) separately.

If $q_1 = [1,p]^T$, $q_2 = [-p,1]^T$ where $p = -A_0^{-1}B_0^{1/2}$, then $C_1(0)q_1 = 0$ and $C_2(0)q_2 = (A^2+B)q_1$. Let $Q = [q_1,q_2]$, then

$$Q^{-1}C_1(0)Q = \begin{bmatrix} 0 & A^2+B \\ 0 & 0 \end{bmatrix} .$$

Let $Q^{-1}C_1(\varepsilon)Q = T = (t_{ij})$ and let

$$M(\varepsilon) = \begin{bmatrix} 1 & 0 \\ t_{11} & t_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & A^2+B \end{bmatrix} + O(\varepsilon).$$

Then for ε sufficiently small, $M(\varepsilon)$ is nonsingular and

$$M(\varepsilon)Q^{-1}C_1(\varepsilon)QM^{-1}(\varepsilon) = \begin{bmatrix} 0 & 1 \\ -\det(T) & \operatorname{trace}(T) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\det(C_1) & \operatorname{trace}(C_1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{bmatrix}$$

by (6.3.26). Let $s = QM^{-1}(\varepsilon)r$, then (6.3.28) becomes

$$\dot{r} = \begin{bmatrix} 0 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{bmatrix} r + M(\varepsilon)Q^{-1}\overline{N}(QM^{-1}(\varepsilon)r\psi + h(s,\varepsilon)). \qquad (6.3.29)$$

To check the hypotheses of Section 9 of Chapter 4 we only need calculate the nonlinear terms when $\varepsilon = 0$. Using the fact that $h(s,0) = O(s^2)$, routine calculations show that

$$M(0)Q^{-1}\bar{N}(QM^{-1}(0)r\psi + h(s,0)) = [p_1,p_2]^T R(r_1,r_2) + O(|r_1^3| + |r_2|^3)$$

where

$$p_1 = \frac{8}{3\pi}(1+p^2)^{-1}(1-p), \quad p_2 = \frac{-8}{3\pi}(1+p^2)^{-1}(1+p)(A^2+B)$$

$$R(r_1,r_2) = \alpha_1 r_1^2 + \alpha_2 r_1 r_2 + \alpha_3 r_2^2$$

$$\alpha_1 = BA^{-1} - 2B^{1/2}, \quad \alpha_2 = 2(A^2+B)^{-1}(B^{3/2}A^{-2}+A-BA^{-1}).$$

Using (6.3.27), $\alpha_1 = (\pi^2 D_0)^{-1}(1-\pi^4 D_0^2)$ so that α_1 is non-zero if $\pi^2 D_0 \neq 1$. We assume that $\pi^2 D_0 \neq 1$ from now on. Note, also that since $1 + p = -(\pi^2 D_0)^{-1}$, we have that p_2 is non-zero.

To reduce (6.3.29) to the form given in Section 9, Chapter 4 we make the substitution

$$\rho = (I - p_1 p_2^{-1} A(\varepsilon))r. \qquad (6.3.30)$$

Substituting (6.3.30) into (6.3.29) and using the above calculations we obtain

$$\dot{\rho} = \begin{bmatrix} 0 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{bmatrix} \rho + F(\rho,\varepsilon) \qquad (6.3.31)$$

$$\alpha = \frac{\partial^2 F_2(0,0)}{\partial \rho_1^2} = \alpha_1 p_2, \quad \beta = \frac{\partial^2 F_2(0,0)}{\partial \rho_1 \partial \rho_2} = 2\alpha_1 p_1 + \alpha_2 p_2$$

$$\frac{\partial^2 F_1(0,0)}{\partial \rho_1^2} = 0.$$

We have already checked that $\alpha \neq 0$. For most values of D_0, β is nonzero [β is only zero when D_0 is a solution of a certain algebraic equation]. If $\beta \neq 0$ then we can apply the theory given in Section 9, Chapter 4 to obtain the bifurcation set for (6.3.23). If $\beta = 0$ then the theory given in Section 9, Chapter 4 still gives us part of the bifurcation set; the full bifurcation set would depend on higher order terms.

Remark. If we vary θ in (6.3.23) the theory given in Section 1 does not apply since the map $(\theta, v) \to \theta v_{xx}$ is not even defined on the whole space. However, it is easy to modify the results of Section 1 to accommodate the above situation. (See, for example, Exercises 1-2 in Section 3.4 of [30].)

REFERENCES

[1] R. Abraham and J. Marsden, Foundations of Mechanics, 2nd ed., Benjamin, New York (1978).

[2] V. Arnold, Bifurcations in versal families, Russian Math. Surveys 27, 54-123(1972).

[3] J.F.G. Auchmuty and G. Nicolis, Bifurcation analysis of nonlinear reaction-diffusion equations, Bull. Math. Biology 37, 323-365(1975).

[4] A.V. Balakrishnan, Applied Functional Analysis, Applications of Mathematics, Vol. 3, Springer-Verlag, Berlin and New York (1976).

[5] J.M. Ball, Stability theory for an extensible beam, J. Diff. Eqns. 14, 399-418(1973).

[6] J.M. Ball, Saddle point analysis for an ordinary differential equation in a Banach space and an application to dynamic buckling of a beam, Nonlinear Elasticity, J. Dickey (ed.), Academic Press, New York (1974).

[7] J.M. Ball, Strongly continuous semigroups, weak solutions, and the variation of constants formula, Proc. Amer. Math. Soc. 63, 370-373(1977).

[8] J.M. Ball and J. Carr, Decay to zerc in critical cases of second order ordinary differential equations of Duffing type, Arch. Rat. Mech. Anal. 63, 47-57(1976).

[9] M. Berger, Nonlinearity and Functional Analysis, Academic Press, New York (1977).

[10] J. Boa and D.S. Cohen, Bifurcation of localized disturbances in a model biochemical reaction, SIAM J. Appl. Math. 30, 123-135 (1976).

[11] J. Carr and Naji-Al-Amood, Rate of decay in critical cases, submitted to the J. Math. Anal. and Appl.

[12] J. Carr and R.G. Muncaster, The application of centre manifolds to amplitude expansions, submitted to SIAM J. Appl. Math.

[13] N.N. Chafee, The bifurcation of one or more closed orbits from an equilibrium point of an autonomous differential system, J. Diff. Eqns. 4, 661-679(1968).

[14] N.N. Chafee, A bifurcation problem for a functional differential equation of finitely retarded type, J. Math. Anal. Appl. 35, 312-348(1971).

[15] S.N. Chow and J. Mallet-Paret, Integral averaging and bifurcation, J. Diff. Eqns. 26, 112-159(1977).

[16] S.N. Chow and J. Mallet-Paret, Fuller index and Hopf bifurcation, J. Diff. Eqns. 29, 66-85(1978).

[17] S.N. Chow, J. Mallet-Paret, and J.K. Hale, Applications of generic bifurcation, I, Arch. Rat. Mech. Anal. 59, 159-188(1975); II, Ibid. 62, 209-235(1976).

[18] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York (1955).

[19] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, Indiana Univ. Math. J. 21, 193-226(1971).

[20] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, J. Diff. Eqns. 31, 53-98(1979).

[21] M. Golubitsky and D. Schaeffer, A theory for imperfect bifurcation via singularity theory, Commun. Pure Appl. Math. 32, 21-98(1979).

[22] M. Golubitsky and D, Schaeffer, Bifurcation analysis near a double eigenvalue of a model chemical reaction, preprint.

[23] J.K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York (1969).

[24] J.K. Hale, Critical cases for neutral functional differential equations, J. Diff. Eqns. 10, 59-82(1970).

[25] J.K. Hale, Lecture on generic bifurcation theory, Symposium on Nonlinear Analysis and Mechanics, Vol. I, R.J. Knops (ed.), *Lecture Notes in Mathematics*, Pitmann, New York (1977).

[26] P. Hartman, *Ordinary Differential Equations*, Second Edition, P. Hartman (1973).

[27] P. Hassard and Y.H. Wan, Bifurcation formulae derived from center manifold theory, J. Math. Anal. Appl. 63(1), 297-312(1978).

[28] A.R. Hausrath, Stability in the critical case of pure imaginary roots for neutral functional differential equations, J. Diff. Eqns. 13, 329-357(1973).

[29] F.G. Heinenken, H.M. Tsuchiya and R. Aris, On the mathematical status of the pseudo-steady state hypothesis of biochemical kinetics, Math. Biosci. 1, 95-113(1967).

[30] D. Henry, Geometric Theory of Parabolic Equations, to be published by Springer-Verlag.

[31] M. Hirsch, C. Pugh and M. Schub, Invariant Manifolds, Springer Lecture Notes in Math., Vol. 585, Springer-Verlag, Berlin and New York (1977).

[32] P.J. Holmes, Bifurcations to divergence and flutter in flow induced oscillations - a finite dimensional analysis, J. Sound and Vib. 53, 471-503 (1977).

[33] P.J. Holmes, Coupled flutter and divergence in autonomous nonlinear systems, to appear in A.S.M.E. *Symposium* on Nonlinear Dynamics.

[34] P.J. Holmes and J.E. Marsden, Bifurcations to divergence and flutter in flow induced oscillations - an infinite dimensional analysis, Automatica 14(4), 367-384(1978).

[35] E. Hopf, Abzweigung einer periodischen Losung von einer stationaren Losung eines Differential systems, Ber. Math. Phys., Kl. Sachs Akad. Wiss. Leipzig 94, 1-22(1942).

[36] M.C. Irwin, On the stable manifold theorem, Bull. London Math. Soc. 2, 196-198(1970).

[37] A. Kelley, The stable, center-stable, center, center-unstable, and unstable manifolds, J. Diff. Eqns. 3, 546-570(1967).

[38] A. Kelley, Stability of the center-stable manifold, J. Math. Anal. Appl. 18, 336-344(1967).

[39] N. Kopell and L.N. Howard, Bifurcations and trajectories joining critical points, Adv. Math. 18, 306-358(1976).

[40] O.E. Langford, Bifurcation of Periodic Solutions into Invariant Tori: The Work of Ruelle and Takens Springer Lecture Notes in Mathematics, Vol. 322, pp. 159-192, Springer-Verlag, Berlin and New York (1972).

[41] J.E. Marsden, Qualitative methods in bifurcation theory, Bull. Amer. Math. Soc. 84, 1125-1147(1978).

[42] J.E. Marsden and M.F. McCracken, The Hopf Bifurcation and Its Applications, Applied Math. Sciences, Vol. 19, Springer-Verlag, Berlin and New York (1976).

[43] S.J. Merrill, A model of the stimulation of B-cells by replicating antigen, I., Math. Biosci. 41, 125-141(1978); II. Ibid, 41, 143-155(1978).

[44] A. Pazy, Semi-groups of linear operators and applications to partial differential equations, University of Maryland Lecture Notes, Vol. 10, (1974).

[45] D. Ruelle and F. Takens, On the nature of turbulence, Comm. Math. Phys. 20, 167-192(1971).

[46] D.H. Sattinger, Topics in Stability and Bifurcation Theory, Springer Lecture Notes in Math., Vol. 309, Springer-Verlag, Berlin and New York (1973).

[47] J.T. Schwartz, Nonlinear Functional Analysis, Gordon and Breach Science Publishers, New York (1969).

[48] S.J. van Strien, Center manifolds are not C^∞, Math. Z. 166, 143-145(1979).

[49] F. Takens, Forced Oscillations and Bifurcations, Communication 3, Math. Institute, Rijksuniversiteit, Utrecht (1974).

[50] F. Takens, Singularities of vector fields, Publ. Inst. Hautes Etudes Sci. 43, 47-100 (1974).